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- Theoretical and practical interest of the concept of 'graph'
- Ordered pairs
- Elements having a specified property
- Looking for an optimal path
- How saturation arises
- Important concepts associated with a lattice
- Ignoring the sense of the arrows
- Important generalisations of the concept of 'graph'
- From the concept of 'graph' to that of 'image'

The theory of graphs provides a basic language and technique by means of which a wide variety of problems can be analysed and solved. These problems arise not only in the pure and applied sciences but also in fields such as sociology, economics, psychology, linguistics, game theory, and so on.

This book provides a general introduction to the theory of graphs presented in a manner which should appeal to the reader who, whilst wanting his knowledge and understanding based on a firm footing, is nevertheless not concerned at this stage with long and detailed proofs of theorems. The Author is careful to give the theory a firm basis in modern algebra, but much of the material would still be of interest to the reader whose knowledge of sets is limited and who might therefore wish to omit making a detailed study of the purely algebraic sections of the book. There are frequent references to a wide variety of applications of graph theory, including the search for optimal paths, and there is a fascinating glimpse into recent and possible future applications in the field of pattern recognition.

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POINTS AND ARROWS

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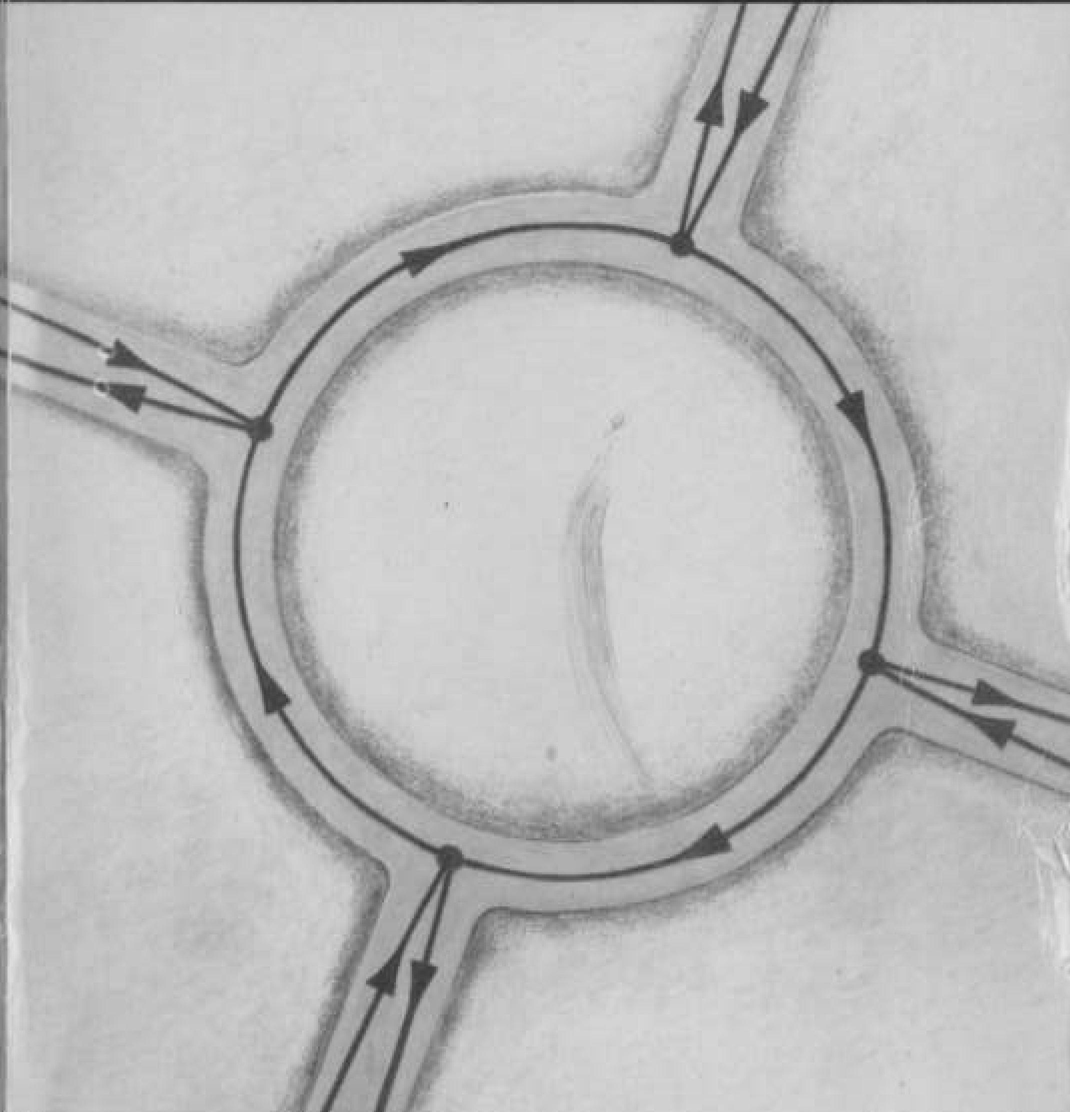
POINTS AND ARROWS

THE THEORY OF GRAPHS

A Kaufmann



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Points and Arrows: the Theory of Graphs

A. KAUFMANN

This is a translation of *Des points et des flèches: la théorie des graphes*, originally published by Dunod, Paris, in 1968. It provides a general, non-specialised introduction to the theory of graphs. After some initial remarks on the wide variety of applications of the theory, the author introduces the concept of *graph* and puts it in its basic set theory context. A number of practical problems and methods are then discussed, including latin sequences, optimal paths and the travelling salesman problem. Some further theoretical concepts are developed, including those of non-directed graph, lattice, tree, together with the associated problems. The book concludes with a brief look at the developing techniques of the study of pattern recognition.

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Reader in Mathematics, The Open University

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Points and Arrows:
the Theory of Graphs

A. Kaufmann

Professor at l'Institut polytechnique de Grenoble

Translated by
H. Graham Flegg

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Contents

Editor's Foreword

The last twenty years or so have seen a remarkable development in the study of finite combinatorial problems. These problems arise in a very wide variety of situations and are of pressing interest not only in the pure and applied sciences but also in fields such as sociology, economics, psychology, linguistics, games, and so on. The *theory of graphs* provides a basic language and technique by means of which such problems can be isolated, analysed and exhaustively investigated. A few extensive works devoted to graph theory are now generally available, but, for the most part, these have been written with the requirements of the specialist in mind, and there has for some time been a real need for a brief introductory book written for popular consumption, without long and detailed proofs of theorems, as a whetter of the appetite and a pointer towards possible developments of the theory rather than a complete and rigorous textbook.

This little book by Professor Kaufmann, which first appeared in the French language in 1968, fulfils just such a function, and it is particularly fitting, therefore, that an English translation of it should be one of the first group of works to appear in the *Transworld Student Library*. The book is conceived in a manner which should appeal to the more general reader, but Professor Kaufmann is careful to give the theory a firm basis in contemporary elementary mathematics whilst at the same time stimulating the reader by frequent references to a wide variety of applications and an enticing glimpse into recent and possible future developments. In preparing and editing this translation, I have been forcibly reminded of the way in which yet another modern branch of mathematical study can provide an under-

lying unifying basis for a wide variety of pure and applied disciplines. The theory of graphs seems likely to be of increasing importance as the scientific study of *decision-making* continues its development over the next decade or so, and some familiarity with its terminology and principles is likely to be an essential part of the intellectual and practical equipment of the scientist, the sociologist, the psychologist, the economist, etc., of the immediate future.

H. GRAHAM FLEGG

1. The Theoretical and Practical Interest of the Concept of 'Graph'

The theory of graphs is simply that branch of set theory which concerns the binary relations of a set with itself, the set being countable. This theory is not independent of other developments in set theory, but it possesses its own extensive and specialised vocabulary, and it covers a very large field of applications in physics, in economics, in psychology, in telecommunications, in organic chemistry, in management, in certain aspects of art, and so on.

With the intention of apprising the reader of various fields of applications, we now give a list of problems which are studied and solved with the aid of the theory of graphs.

Scheduling and Sequencing

A manufacturing process consists of a certain number of operations, A, B, C, ... , which can follow each other in a designated order but subject to technological constraints. To obtain a solution, or better still to obtain an optimal solution relative to some prescribed function, is an important problem facing managers responsible for productivity.

Another kind of problem of scheduling and sequencing relates to networks established according to the principles of PERT (Program Evaluation and Review Technique). The execution of a construction programme involves delays between different operations; we wish to be informed of the allowable margins at the start of the carrying out of each operation; we

also wish to know the date before which the complete programme of work cannot be finished. A sequence of critical operations or 'critical path' is defined, on which all the information controlling the delays must be recorded. Such a problem amounts to that of a search for a greatest value path in a graph. PERT is widely used, particularly in public works and buildings, in research programmes, and in large scientific and industrial enterprises, in the film industry, . . . right up to solicitors who are now using it for very complicated legal and administrative operations.

Traffic and Transport

Urban highway traffic flow arises in the form of directed networks where vehicles are travelling in large numbers. These networks are precisely 'graphs' in the sense of the word as we use it here. A network of railway lines, a network of airways, any complex system of communications is also a graph. The distribution of goods by trains or lorries or ships, the circulation of documents, and so on, constitute also combinatorial structures which can be studied conveniently using the concept of 'graph'.

Computers

A computer is, before all else, an extremely complex electrical network, which is a graph if one considers the way in which the circuits and electronic elements which go to make it up are interconnected. The brain of an animal, and particularly that of a man, is a graph in which around 10^9 neurons and more than 10^{12} separate connections between them can be found.

Organic Chemistry

In organic chemistry the body can be represented using non-directed graphs or, more precisely, using multi-graphs in which the concept of chemical valency plays a fundamental role.

The theory of graphs also arises in the classification of known bodies and in research into hypothetical bodies.

Communications in a Human Group. Psychosociology

The structure of networks of communications between individuals is of interest here, and in these networks one studies various symbols and numerical scales especially significant in psychosociology. Centralisation and its effects, the dangers of certain critical positions in a network where temporary absence can throw it into disorder, the security of communications, and so on, are important aspects of such studies. The psychologist and the manager can obtain great benefit from use of the theory of graphs.

Textual Criticism. Documentary Research

The philologist and the student of maps wish to determine the validity of historical texts which have been subjected to many alterations when copies have been made over the years. The classical methods of textual criticism can be appreciably improved upon, and can therefore be applied in cases much more complicated than is normal, by means of certain concepts of the theory of graphs. All the continuities and the mishaps which have occurred (or are supposed to have occurred) form a lattice structure, a concept of relation in an ordered set which can be represented by a graph.

As for the problems arising out of documentary research, so important in the modern world, these are complicated problems of classification, of sorting, of selection, of comparison, whose structures give rise to graphs.

Rules of Certain Games

The theory of games of strategy, of which the game of chess is an example, but of which the military applications are very

important (and even in some cases various civilian applications), constitute an area of study where several theorems on graphs play an essential role.

Problems of Flow in a Network

The flows (or flux) of matter in a network, allowing continuous passage of material, often constitute problems to be found in very varied forms. The matter under consideration can be made up of movements, of pieces of information, of positions of elements; the matter in such cases being abstract as well as concrete.

Markov Chains. Processes of Chance

One of the more important ideas in the theory of probability is that of processes of chance, where a system develops by passing from one state to another or retains its original state according to certain conditions specified by the laws of probability. In the case where the states form a countable set, one often encounters a concept called a *Markov chain*. The mathematical treatment of these Markov chains is considerably simplified by using the concept of a graph.

Population Growth. Dynamic Demography

The evolution of human groups poses important problems, particularly for the provision of the means whereby the social needs of these populations can be satisfied. For example, the problem of the planning of school facilities (premises, teachers, canteens, transport, etc.) for a population of school-children which changes annually in distribution can be more easily defined using the terminology of graphs.

The Concept of a Graph in Aesthetics

The relative positions of objects, the proximities of colours, are questions which are treated with the aid of graphs by those

investigators and artists who interest themselves in pictorial structure.

As for musical composition, it can be shown that it can be carried out by means of rules which are found in developing systems forming Markov chains easily represented by graphs.

Treatment of Linguistic Information

In this area of study, the problems of global recognition of form and structure are typically problems in the theory of graphs when one considers their combinatorial aspects.

Other important questions involve the concept of a graph: matrices of apprenticeship, the senses, associative memory, and so on.

2. Ordered Pairs

The Concept of Set

In mathematics, a collection of well defined objects, each one different from the others, is called a *set*. We can speak of the set of letters of the alphabet, the set of points of a plane, the set of whole numbers; we cannot speak of the set of characters forming the words on this page (these occur more than once), or of the set of ideas (these are not exactly specified). When the elements forming a set can be counted, the set is said to be *countable*; otherwise, the set is said to be infinitely *non-countable*. If, when we count the elements, we come to a stop, the set is said to be *finite*. For example, the set of inhabitants of a town is finite (though in visiting India or China one may believe that it is nothing of the kind). In this book, we shall consider finite sets only, but the concepts and properties developed as we go along can all be applied also in the case of infinite countable sets; we shall not do this here as it would be appreciably more complicated.

So, let E be a finite set, and let us consider all the ordered pairs (X_i, X_j) which can be formed from the elements of E, but bearing in mind beforehand that an *ordered pair* is an ordered collection of two elements, whereas a *pair* is an unordered collection of two elements. If an element X_i belongs to a set E we write $X_i \in \mathsf{E}$; if it does not belong to E we write $X_i \notin \mathsf{E}$. The set of all ordered pairs (X_i, X_j) is called the *product set*, denoted by $\mathsf{E} \times \mathsf{E}$; we can therefore write $(X_i, X_j) \in (\mathsf{E} \times \mathsf{E})$.

Consider a set, say,

(1) $$\mathbf{E} = \{A, B, C, D\}.$$

It is easy to construct the set of all ordered pairs of elements of $\mathbf{E}$, namely,

(2) $$\begin{aligned}\mathbf{E} \times \mathbf{E} = \{&(A, A), (A, B), (A, C), (A, D),\\ &(B, A), (B, B), (B, C), (B, D),\\ &(C, A), (C, B), (C, C), (C, D),\\ &(D, A), (D, B), (D, C), (D, D)\}.\end{aligned}$$

We can use various methods of representing the set of these ordered pairs: we can use a grid (figure 1), points and arrows (figure 2), connected points (figure 3), or we can use a matrix (figure 4).

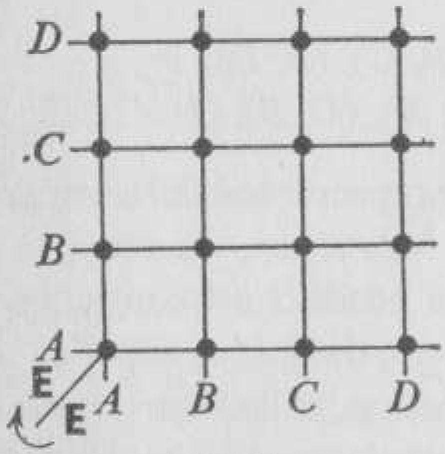


Fig. 1

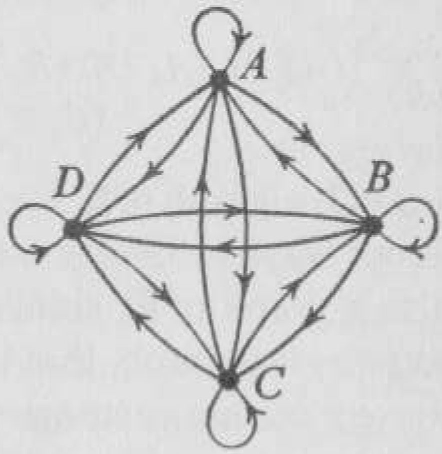


Fig. 2

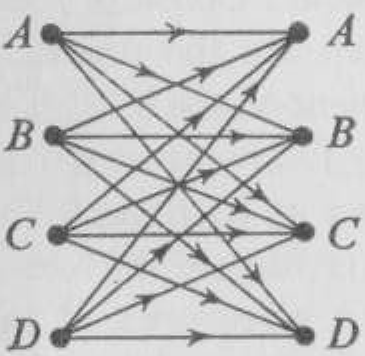


Fig. 3

	A	B	C	D
A	1	1	1	1
B	1	1	1	1
C	1	1	1	1
D	1	1	1	1

Fig. 4

Consider now a set $\mathbf{E}$ and the product set $\mathbf{E} \times \mathbf{E}$ associated with it. Suppose that certain ordered pairs possess a property $\mathscr{P}$ whilst others do not possess this property; we shall denote this by saying that the other ordered pairs have the property not-$\mathscr{P}$ which we shall write as $\bar{\mathscr{P}}$. Thus, the property $\mathscr{P}$ effects a bi-partition (the set is divided into two parts having no elements in common). The definition of the concept of graph is directly linked with this idea of bi-partition. If we make a bi-partition of a set $\mathbf{E} \times \mathbf{E}$, and if we distinguish the ordered pairs belonging to one of the parts from those belonging to the other, we form a graph. A printed picture is a graph; it is made up of black points and white points. A line drawn on a white sheet of paper is a graph. The word 'graph', looking to Greek etymology, is a termination rather like '—gram'.

Consider the following subset of $\mathbf{E} \times \mathbf{E}$:

$$(3) \quad \mathbf{G} = \{(A, B), (A, D), (B, B), (B, C), (B, D), \\ (C, C), (D, A), (D, B), (D, C), (D, D)\}.$$

This subset is a graph of $\mathbf{E}$. We have represented the same graph in different ways in figures 5 to 8. Moreover, the set $\mathbf{E} \times \mathbf{E}$ is itself also a graph of $\mathbf{E}$; similarly, a product set containing no ordered pairs is a graph, that is to say, the empty set of $\mathbf{E} \times \mathbf{E}$.

We have a certain customary notation, called the notation of *correspondence* or *multivocal* (i.e. *many-many*) *mapping*, though strictly a mathematician should not associate the word 'mapping' with the adjective 'multivocal'. Consider the graph represented by figures 5 to 8. Which are the ordered pairs having A as the first element (the element on the left)? They are (A, B) and (A, D). If we are at A, by following an arrow corresponding to an ordered pair, we can go to B or D. We say that the multivocal mapping of A is formed by the subset $\{B, D\}$. If we are at B, by following a given ordered pair, we can go to B, C, or D. We say that the multivocal mapping of B

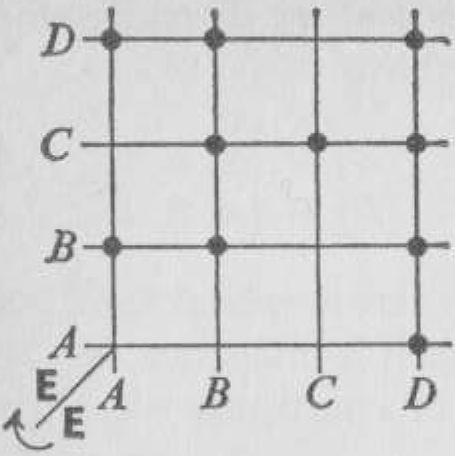


Fig. 5

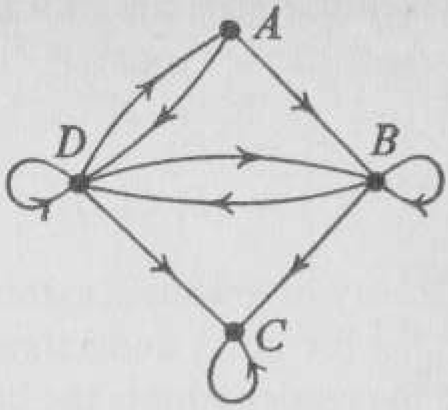


Fig. 6

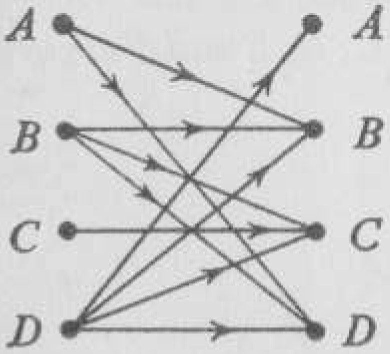


Fig. 7

	A	B	C	D
A	0	1	0	1
B	0	1	1	1
C	0	0	1	0
D	1	1	1	1

Fig. 8

is $\{B, C, D\}$, and so on. Using the symbol Γ to indicate the multivocal mapping of all the elements of the graph, we write:

$$\Gamma\{A\} = \{B, D\}, \qquad \Gamma\{B\} = \{B, C, D\},$$
$$\Gamma\{C\} = \{C\}, \qquad \Gamma\{D\} = \{A, B, C, D\}. \tag{4}$$

If one of the elements has no other element corresponding to it, that is to say, if, looking at the representation by points and arrows, in leaving a certain element X_i we are unable to go to any other element, we will write $\Gamma\{X_i\} = \varnothing$, where this symbol represents the empty set. (In operations with sets, this symbol plays a role similar to that of 0—zero—in operations with numbers.)

We consider also the inverse multivocal mapping denoted by

Γ^{-1}, which corresponds to a reversal of the direction of each arrow.

$$(5) \qquad \Gamma^{-1}\{A\} = \{D\}, \qquad \Gamma^{-1}\{B\} = \{A, B, D\},$$
$$\Gamma^{-1}\{C\} = \{B, C, D\}, \qquad \Gamma^{-1}\{D\} = \{A, B, D\}.$$

The theory of graphs is extremely rich in special terms which are needed for good understanding. It is important that from now on the reader adopts the habit of a particular vocabulary.

Vertex

An element of $\mathbf{E}$ in the graph $\mathbf{G}$ is called a *vertex*. Thus, in the graph of figure 9, the elements A, B, C, D, E and F are *vertices*. We also call them *points*.

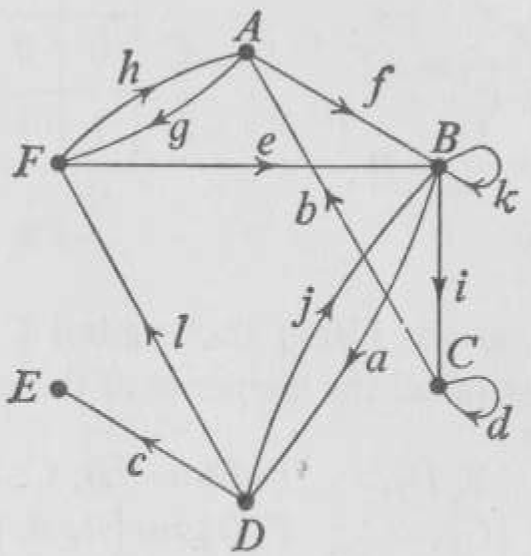


Fig. 9

Arc

An element of $\mathbf{G}$ is called an *arc*. Thus, in the graph of figure 9, the elements (A, B), (A, F), (B, B), (B, C), (B, D), (C, A), (C, C), (D, B), (D, E), (D, F), (F, A), and (F, B) are *arcs*. On the other hand, the ordered pairs (A, D) and (E, C), for example, are not arcs, nor are (A, C) or (B, A).

Berge's Notation

Professor Berge uses the following notation, which we shall now adopt.

A graph will be designated by an ordered pair formed from the set of vertices $\mathbf{E}$ together with the multivocal mapping Γ which defines the correspondence of the elements. Thus, a graph will be designated by

$$G = (\mathbf{E}, \Gamma). \tag{6}$$

Always in using Berge's notation, if $\mathbf{U}$ designates the set of arcs, a graph will also be designated by

$$G = (\mathbf{E}, \mathbf{U}). \tag{7}$$

Thus, in the example of figure 9:

$$\mathbf{U} = \{(A, B), (A, F), (B, B), \ldots, (F, B)\}. \tag{8}$$

The arcs are often designated by their own special letters (see figure 9).

Partial Graph of a Graph

If we retain all the vertices but delete one or more of the arcs, we obtain a *partial graph* of the given graph.†

Thus, G_1 (figure 11) is a partial graph of G (figure 10).

Sub-graph of a Graph

If we remove one or more vertices together with the arcs having one end or both ends at these vertices, the graph obtained is called a *sub-graph* of the given graph.‡

†The definition is extended to the case where no arc is removed. Thus a graph is a partial graph of itself.

‡The definition is extended to the case where no vertex is removed. Thus a graph is a sub-graph of itself.

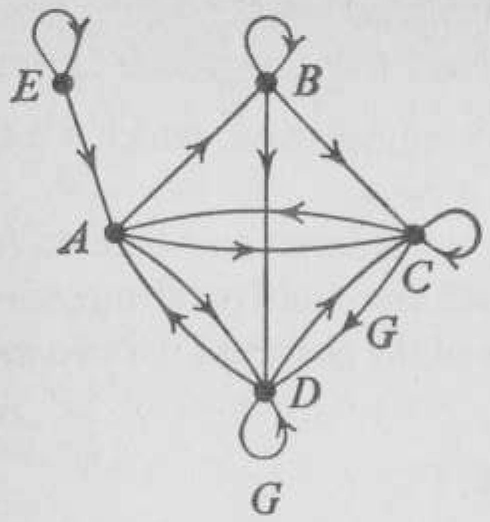


Fig. 10

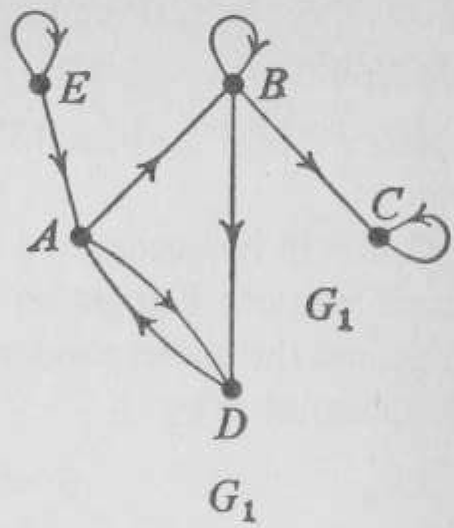


Fig. 11

Thus G_2 (figure 12) is a sub-graph of G (figure 10).

We can consider *partial sub-graphs.* Thus G_3 (figure 13) is a partial graph of G_2 (figure 12) and a partial sub-graph of G (figure 10).

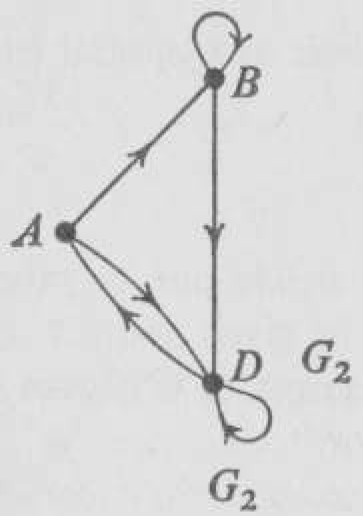


Fig. 12

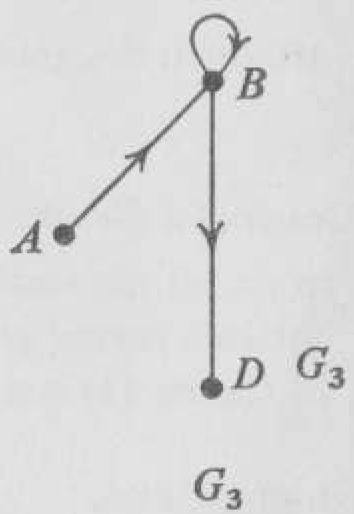


Fig. 13

Symmetric Graph

A graph is *symmetric* if the existence of an ordered pair (X, Y) belonging to the set $\mathbf{U}$ is always accompanied by the existence of the ordered pair (Y, X). Thus, the graph of figure

14 is symmetric. In a representation such as those of figures 5 and 8 a symmetric graph gives rise to a table which is symmetric about the principal diagonal.

Antisymmetric Graph

A graph is *antisymmetric* if the existence of an ordered pair (X, Y) belonging to **U** forbids the existence of the ordered pair (Y, X). Thus the graph of figure 15 is antisymmetric. An antisymmetric graph never contains arcs (X, Y), where $X = Y$ (the graph does not have loops).

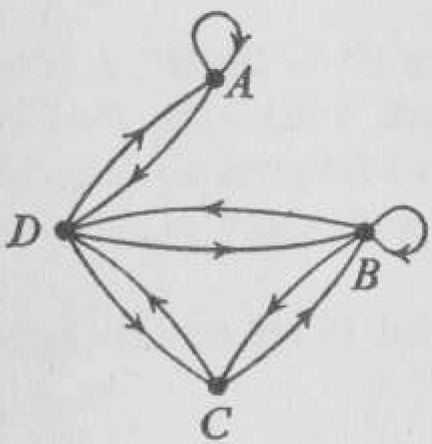


Fig. 14

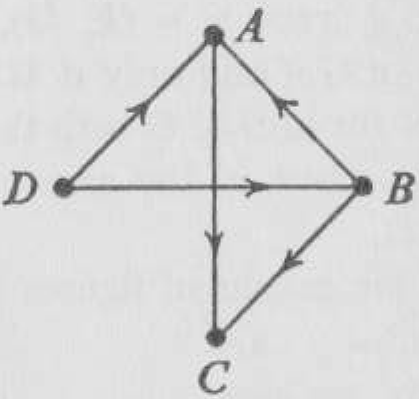


Fig. 15

Complete Graph

A graph is *complete* if, whenever there are vertices X and Y belonging to **E**, at least one of the ordered pairs (X, Y) and (Y, X) belong to **U**, with the possible exception of ordered pairs like (X, X); see figure 16.

Full Graph

A graph is 'full' if all the ordered pairs of its vertices are arcs. Thus the graph of figure 17 is full. A full graph is thus the set $\mathbf{E} \times \mathbf{E}$.

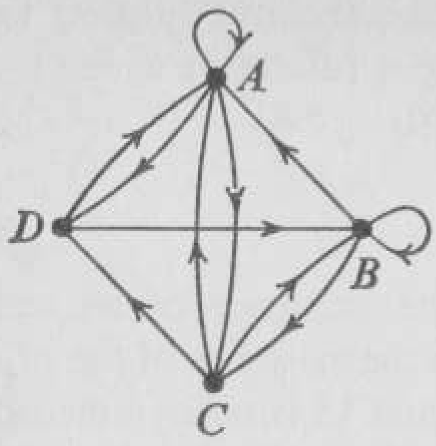


Fig. 16

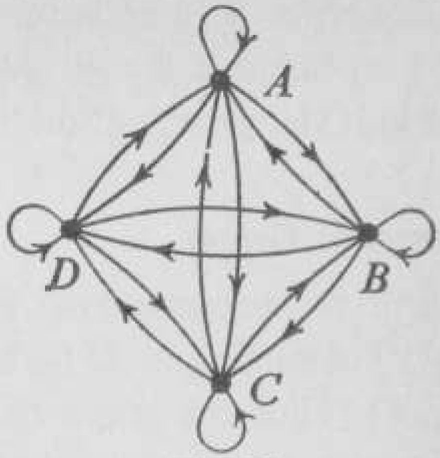


Fig. 17

Complementary Graph

Given a graph $G = (\mathbf{E}, \mathbf{U})$, then $G^* = (\mathbf{E}, \mathbf{U}^*)$ is the *complement* of G if and only if $\mathbf{U}^* = (\mathbf{E} \times \mathbf{E}) - \mathbf{U}$. Further, the union of the arcs of G with those of G^* gives a full graph and the intersection of the arcs of G with those of G^* gives the empty set.

Thus the graphs of figures 18 and 19 are mutually complementary.

Clearly, we have

$$(G^*)^* = G. \tag{9}$$

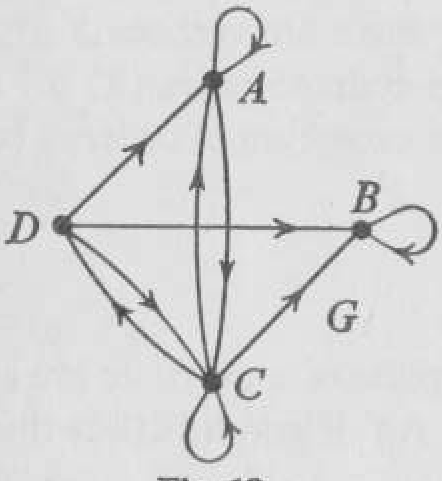


Fig. 18

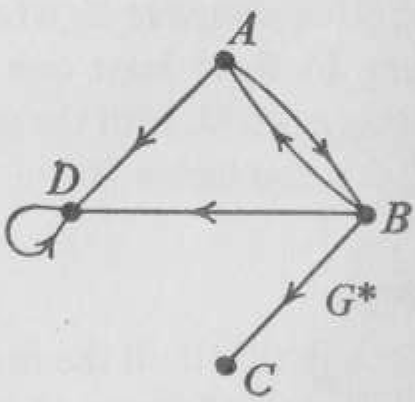


Fig. 19

Path

A *path* is a sequence $(u_1, u_2, u_3, \ldots)$ of arcs such that the final extremity of each arc of the sequence corresponds to the initial extremity of the next one. A path may or may not be finite.

Thus, in figure 20, (a, c, m, i), (f, m, d, b, a), (g, h, n, j, h, q), (k), and (k, c, m, d) are paths. We can also designate a path by the vertices which it contains: (C, A, B, E, D), (C, B, E, A, C, A).

A path is *simple* if it does not contain the same arc twice. Thus (see figure 20), the path (a, c, m, i) is simple and (g, h, n, j, h, q) is not.

A path is *elementary* if it does not utilise the same vertex twice. Thus (see figure 20), the path (a, c, m, i) is elementary, (h, d, c, m, q) is not elementary but simple, and (g, h, n, j, h, q) is not elementary. An elementary path is always simple but the converse is not true.

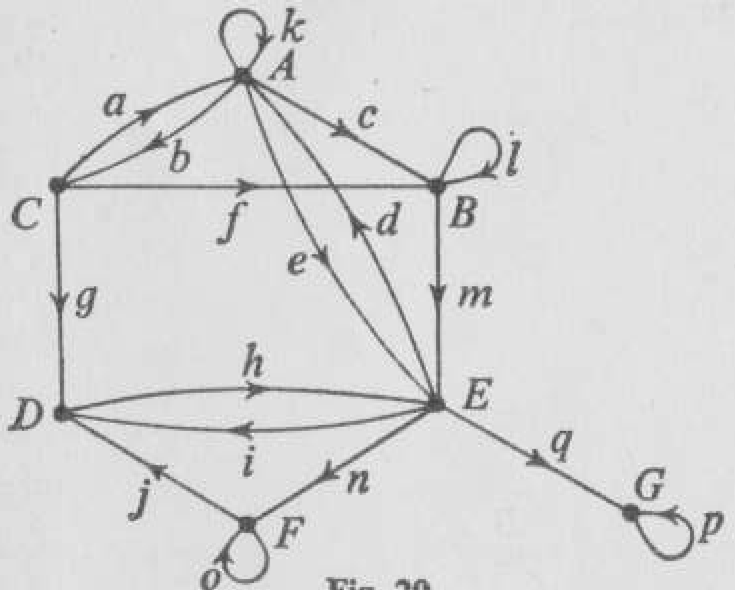


Fig. 20

Circuit

A *circuit* is a finite path where the initial vertex coincides with the final vertex. A circuit can be represented by its vertices

or by its arcs. Thus (in figure 20), (A, B, E, A), (C, A, B, B, E, A, C) and (d, b, g, h) are circuits. The qualifications 'simple' and 'elementary' apply to circuits. A circuit consisting of just two identical vertices is called a *loop*. (F, F) is a loop.

Length of a Path

The *length* of a path is the number of arcs which it contains. If μ is the path, its length is denoted by $l(\mu)$. For example (see figure 20):

$$\mu = (A, B, E, F, D),$$
$$l(\mu) = l(A, B, E, F, D) = 4;$$
$$l(A, B, E, A) = 3$$
$$l(F, F) = 1$$

We agree to give the length 0 to each path containing 1 vertex and 0 arc.

3. Adjacency and Order

Strongly Connected Graph

A graph is *strongly connected* if there exists at least one path from each vertex to every other vertex. For example, the graph of figure 21 is strongly connected; we can check that there is at least one path from each vertex to every other vertex. This is not the case with the graph of figure 22; for example, there is no path from X_4 to X_1.

The concept of being strongly connected is very important in the theory of graphs; we shall see later what *connectivity* (connected graphs) means; this is a weaker concept.

Circuitless Graph

As its name shows, such a graph does not contain a circuit. For example, the graph of figure 23 has no circuit.

This concept plays equally a very important role in the theory we are presenting.

Review of Some Important Properties

We are going to review for the reader several generally well-known concepts, but, having given their fundamental features in what follows, it is advisable that the reader keeps them fresh in his memory.

Every binary relation in a product set $\mathbf{E} \times \mathbf{E}$, where $\mathbf{E}$ is finite, can be represented by a graph.

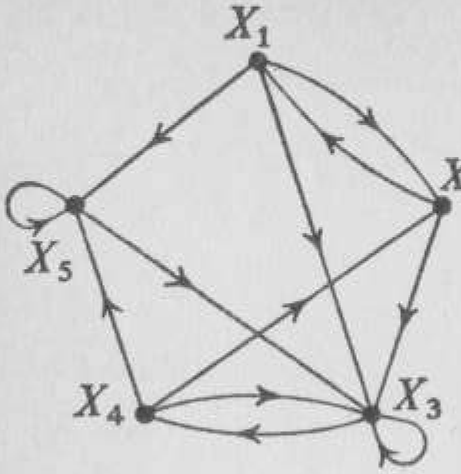


Fig. 21

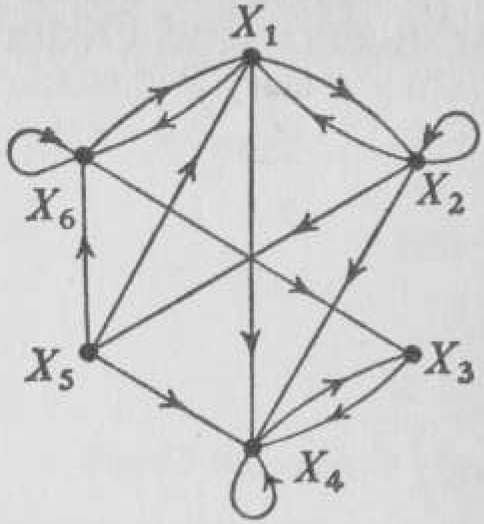


Fig. 22

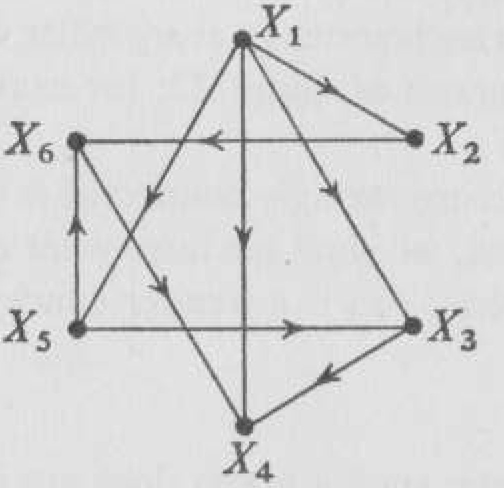


Fig. 23

For convenience of writing, we shall designate the set of ordered pairs representing the binary relation by $\mathbf{G} \subset \mathbf{E} \times \mathbf{E}$.

Reflexive Binary Relation

A binary relation is *reflexive* if:

$$\forall x \in \mathbf{E}: \qquad (x, x) \in \mathbf{G}. \tag{10}$$

In other words, every ordered pair (x, x) belongs to the binary relation. The binary relation represented in figure 24 is reflexive.

Symmetric Binary Relation

A binary relation is *symmetric* if:

(11) $$((x, y) \in \mathbf{G}) \Rightarrow ((y, x) \in \mathbf{G}).$$

In other words, if an ordered pair (x, y) belongs to the relation, the reverse pair also belongs to the relation. For example, see figure 25.

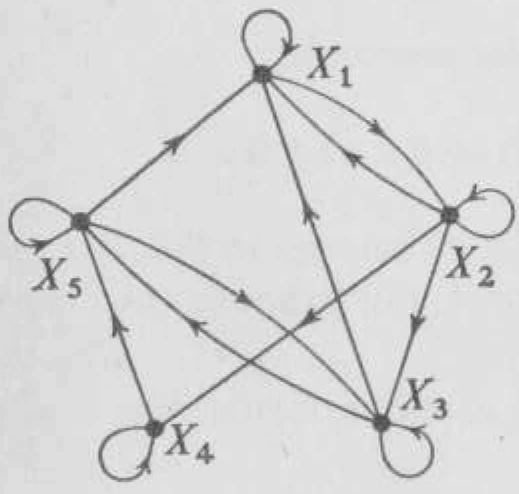


Fig. 24

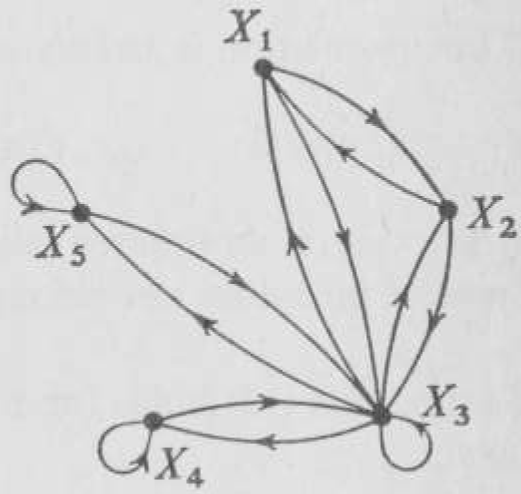


Fig. 25

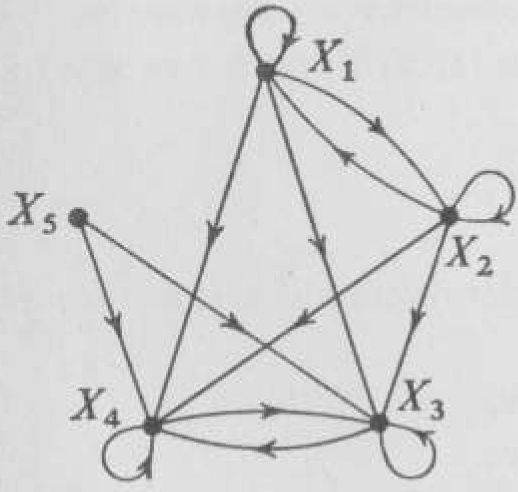


Fig. 26

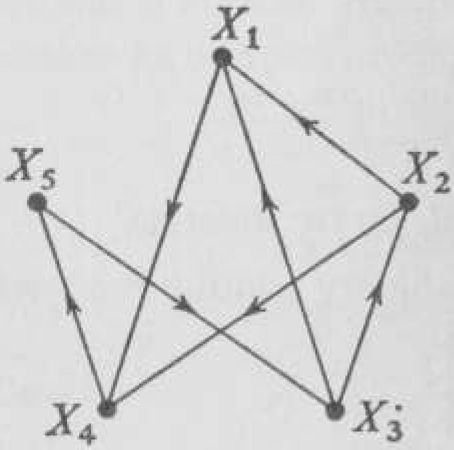


Fig. 27

Transitive Binary Relation

A binary relation is *transitive* if:

$$(12) \qquad ((x, y) \in \mathbf{G}) \quad \text{and} \quad ((y, z) \in \mathbf{G}) \Rightarrow ((x, z) \in \mathbf{G}).$$

This expresses the syllogism: if a implies b and b implies c, then a implies c. Figure 26 gives an example of a transitive binary relation.

Antisymmetric Binary Relation

A binary relation is *strictly antisymmetric* if:

$$(13) \qquad \forall (x, y) \in \mathbf{G}: \qquad ((x, y) \in \mathbf{G}) \Rightarrow ((y, x) \notin \mathbf{G}).$$

That is to say, if an ordered pair (x, y) belongs to the relation, the reverse pair does not belong to it. For example, see figure 27.

If expression (13) holds for $x \neq y$, and if, further, for $x = y$ we have:

$$(14) \qquad \forall x \in \mathbf{E}: \qquad (x, x) \in \mathbf{G},$$

the binary relation is said to be *non-strictly antisymmetric*. For example, if we add an ordered pair (x, x) for each x in figure 27, we get figure 28.

Weak Order Relation

A binary relation is a *weak order relation* if it is at the same time:

reflexive,
transitive.

For example, see figure 28.

Equivalence Relation

A binary relation is an *equivalence relation* if it is at the same time:

reflexive,
symmetric,
transitive.

For example, see figure 29.

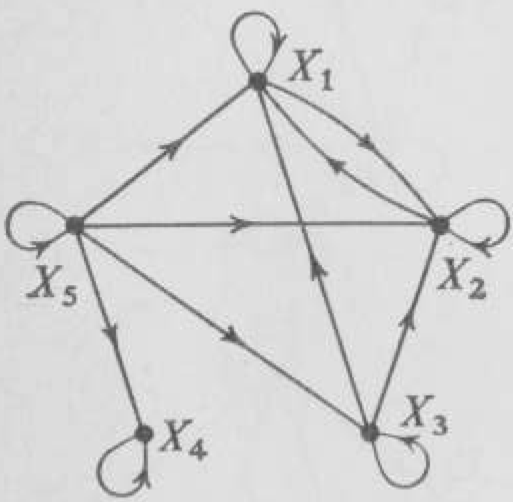


Fig. 28

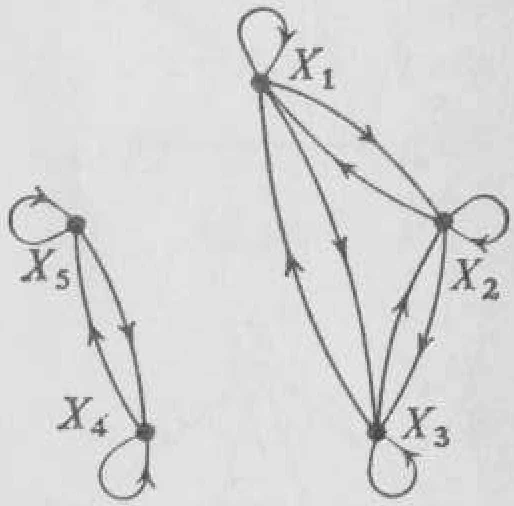


Fig. 29

Strict Order Relation

A binary relation is a *strict order relation* if it is at the same time:

strictly antisymmetric,
transitive.

For example, see figures 30 and 32.

Non-strict Order Relation

A binary relation is a *non-strict order relation* if it is at the same time:

non-strictly antisymmetric,
transitive.

For example, see figures 31 and 33.

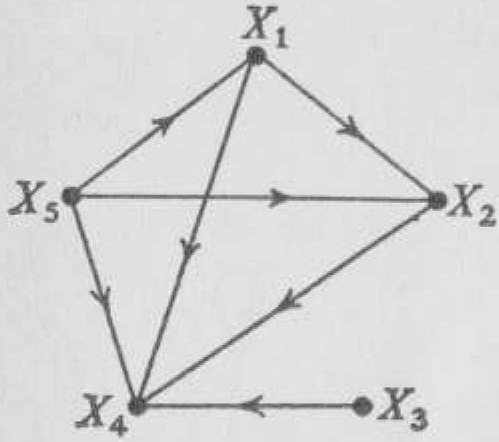


Fig. 30

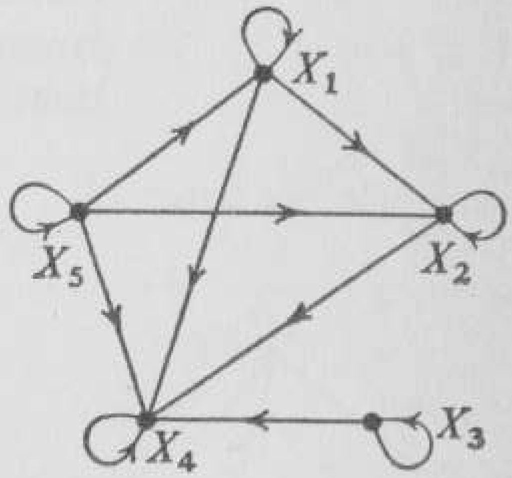


Fig. 31

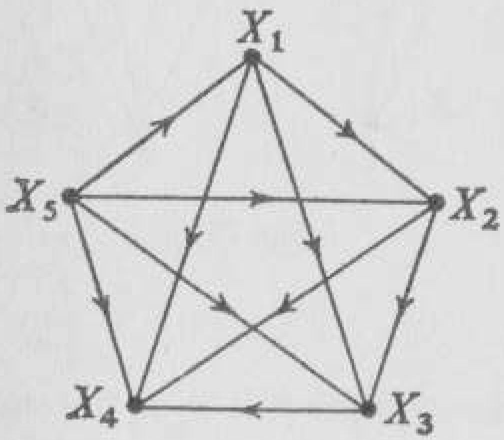


Fig. 32

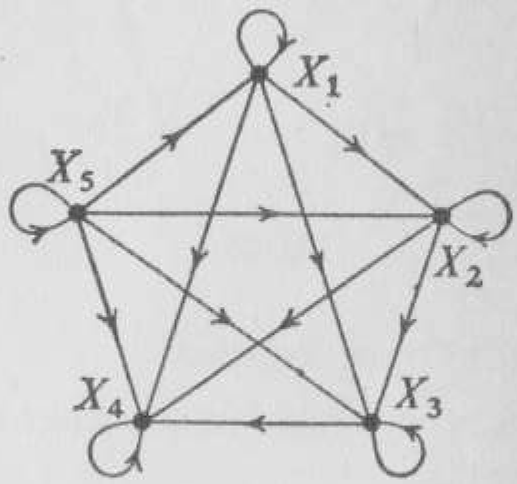


Fig. 33

Total Order Relation and Partial Order Relation

Let us use the symbol $x \prec y$ if x comes before y in the order relation. Now compare figures 30 and 32. For the former we can write: $X_5 \prec X_1 \prec X_2 \prec X_4$ and $X_3 \prec X_4$, but we cannot order X_3 on the one hand with X_5 or, on the other hand, with X_1 or X_2. This is a partial order relation. However, in figure 32

we see that: $X_5 \prec X_1 \prec X_2 \prec X_3 \prec X_4$. This order relation is total. Thus, when all the elements of the set can be ordered, each in relation to every other, the order relation is total; otherwise, it is partial.

We add the qualification 'strict' or 'non-strict' appropriately (figures 30 and 32 in one case, figures 31 and 33 in the other case).

Similarity Relation

A binary relation is a *similarity relation* or an *analogue relation* if it is at the same time:

reflexive,
symmetric.

For example see figure 34.

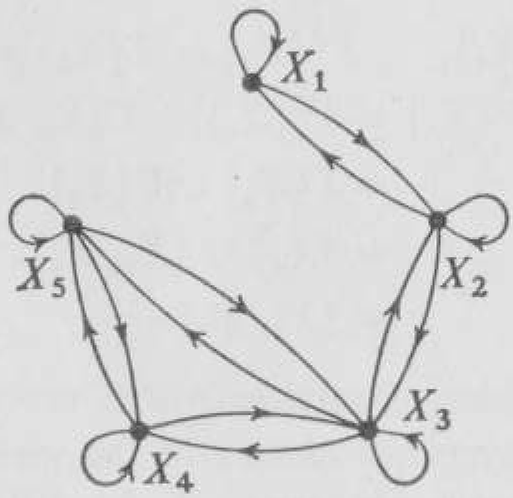


Fig. 34

Having given this review of fundamental concepts in the theory of sets, we return to the theory of graphs.

Transitive Closure of a Vertex

We first define:

$$\Gamma^2\{X_i\} = \Gamma(\Gamma\{X_i\}), \tag{15}$$

then:

$$\Gamma^3\{X_i\} = \Gamma(\Gamma^2\{X_i\}) = \Gamma(\Gamma(\Gamma\{X_i\})), \tag{16}$$

and hence:

$$\Gamma^n\{X_i\} = \Gamma(\Gamma^{n-1}\{X_i\}). \tag{17}$$

Example (see figure 21):

$$\begin{aligned}
\Gamma\{X_1\} &= \{X_2, X_3, X_5\} \\
\Gamma^2\{X_1\} &= \Gamma\{X_2, X_3, X_5\} \\
&= \Gamma\{X_2\} \cup \Gamma\{X_3\} \cup \Gamma\{X_5\} \\
&= \{X_1, X_3\} \cup \{X_3, X_4\} \cup \{X_3, X_5\} \\
&= \{X_1, X_3, X_4, X_5\}.
\end{aligned}$$

Another example (see figure 22):

$$\Gamma\{X_3\} = \{X_4\}, \qquad \Gamma^2\{X_3\} = \Gamma\{X_4\} = \{X_3, X_4\}.$$

$$\begin{aligned}
\Gamma^3\{X_3\} &= \Gamma^2\{X_4\} = \Gamma\{X_3, X_4\} \\
&= \Gamma\{X_3\} \cup \Gamma\{X_4\} \\
&= \{X_4\} \cup \{X_3, X_4\} \\
&= \{X_3, X_4\}.
\end{aligned}$$

We call the subset of vertices which we can reach from a given vertex the *transitive closure* of the vertex; we denote it by $\hat{\Gamma}\{X_i\}$, and it can be expressed, by definition, using the formula:

$$\hat{\Gamma}\{X_i\} = \{X_i\} \cup \Gamma\{X_i\} \cup \Gamma^2\{X_i\} \cup \dots \cup \Gamma^n\{X_i\} \cup \dots. \tag{18}$$

Example (see figure 35):

$$\hat{\Gamma}\{X_2\} = \{X_2\} \cup \Gamma\{X_2\} \cup \Gamma^2\{X_2\} \cup \Gamma^3\{X_2\} \cup \dots,$$
$$\Gamma\{X_2\} = \{X_1, X_2\},$$

$$\begin{aligned}
\Gamma^2\{X_2\} &= \Gamma\{X_1, X_2\} \\
&= \{X_1, X_2, X_4\}, \\
\Gamma^3\{X_2\} &= \Gamma\{X_1, X_2, X_4\} \\
&= \{X_1, X_2, X_3, X_4\}, \\
\Gamma^4\{X_2\} &= \Gamma\{X_1, X_2, X_3, X_4\} \\
&= \{X_1, X_2, X_3, X_4\}, \\
\hat{\Gamma}\{X_2\} &= \{X_2\} \cup \{X_1, X_2\} \cup \{X_1, X_2, X_4\} \cup \{X_1, X_2, X_3, X_4\} \\
&= \{X_1, X_2, X_3, X_4\}.
\end{aligned}$$

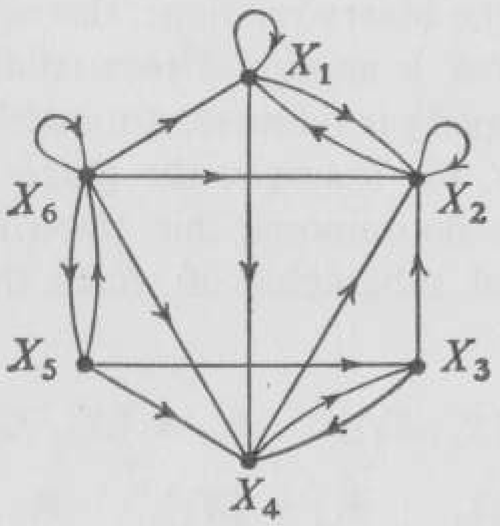


Fig. 35

Let us now define:

$$\begin{aligned}
\Gamma^{-2}\{X_i\} &= \Gamma^{-1}(\Gamma^{-1}\{X_i\}), \\
\Gamma^{-3}\{X_i\} &= \Gamma^{-1}(\Gamma^{-2}\{X_i\}) \\
&= \Gamma^{-1}(\Gamma^{-1}(\Gamma^{-1}\{X_i\})) \\
&\dots\ \dots\ \dots\ \dots\ \dots \\
\Gamma^{-n}\{X_i\} &= \Gamma^{-1}(\Gamma^{-n+1}\{X_i\}), \\
\hat{\Gamma}^{-}\{X_i\} &= \{X_i\} \cup \Gamma^{-1}\{X_i\} \cup \Gamma^{-2}\{X_i\} \cup \dots \\
&\qquad \dots \cup \Gamma^{-n}\{X_i\} \cup \dots .
\end{aligned}$$

This last expression defines what we call the *inverse transitive closure* of the vertex X_i. It is the subset of vertices from which we can reach X_i.

Strongly Connected Maximal Subset

Let us consider any vertex of a graph and the subset of the largest possible number of vertices forming a strongly connected sub-graph together with the given vertex. In this way, we are able to separate the vertices of the graph into classes, these classes forming what we call *equivalence classes* in set theory. Let us show that the binary relation: 'there is a path from X_i to X_j and vice versa' is an equivalence relation. It is easy to verify that the property is reflexive, symmetric and transitive.

Let us consider, for example, the graph of figure 36. In figure 37, we have decomposed this graph into five strongly connected maximal subgraphs, of which the corresponding subsets are:

$$\mathbf{A}_1 = \{X_5, X_7\}, \qquad \mathbf{A}_2 = \{X_2, X_3, X_4\},$$
$$\mathbf{A}_3 = \{X_8\}, \qquad \mathbf{A}_4 = \{X_1\}, \qquad \mathbf{A}_5 = \{X_6\}.$$

This decomposition is unique, but the numbering of the subsets is arbitrary.

We have very simple methods for making such a decomposition. We are going to explain one of them, the very simple principle of which is based on the fact that the equivalence class of a vertex X_i (a vertex of a strongly connected maximal sub-graph) is given by:

$$(19) \qquad \mathscr{C}\{X_i\} = \hat{\Gamma}\{X_i\} \cap \hat{\Gamma}^-\{X_i\}.$$

Let us see how to obtain these classes in the example of figure 37.

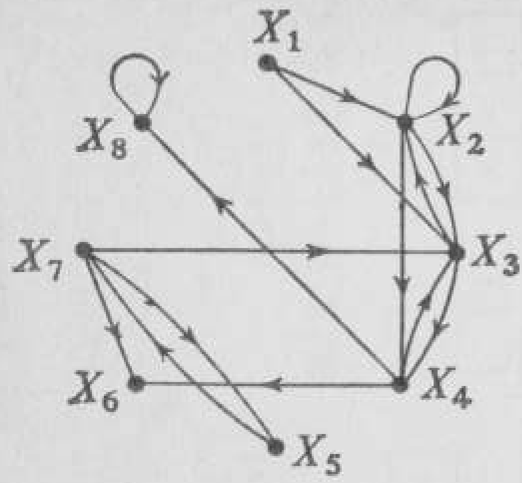


Fig. 36

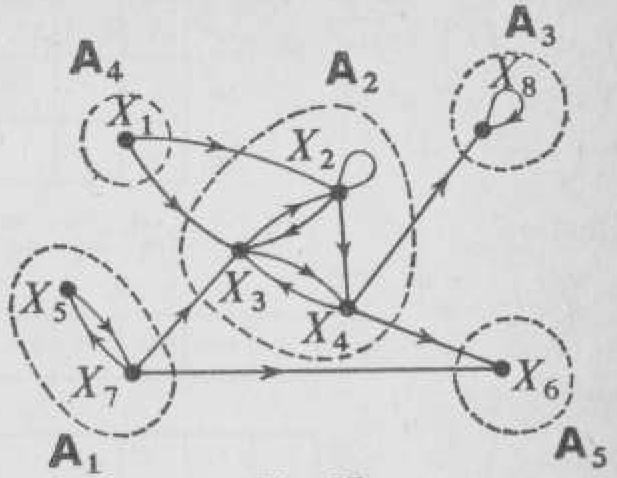


Fig. 37

Let us construct the matrix representation of the graph and put on the right and below, respectively, a column and a row where $\hat{\Gamma}\{X_i\}$ and $\hat{\Gamma}^-\{X_i\}$ will be determined. We make an arbitrary start by looking for the class of $\{X_7\}$ and for $\hat{\Gamma}\{X_7\}$. Opposite to X_7 we put 0 (the shortest path from X_7 to X_7 has length 0). In the X_7 row of the matrix we find the vertices X_3, X_5, X_6; we put 1's in the $\hat{\Gamma}\{X_7\}$ column opposite to rows X_3, X_5, X_6 (this signifies that the shortest path, or paths, from X_7 to X_3, X_5, X_6 have length 1). In rows X_3, X_5, X_6 of the matrix, we find the vertices X_2, X_4, X_7. Vertex X_7 has been dealt with already; opposite rows X_2 and X_4 we put a 2 in the $\hat{\Gamma}\{X_7\}$ column (this signifies that the shortest path, or paths, from X_7 to X_2 or X_4 has the value 2). In rows X_2 and X_4 we find the vertices X_2, X_3, X_4, X_6, X_8. Only the vertex X_8 has not been dealt with; we put a 3 in the $\hat{\Gamma}\{X_7\}$ column in the X_8 position (this signifies that the shortest path, or paths, from X_7 to X_8 has length 3). As we find only X_8 in the X_8 row, we can go no further. Only the vertex X_1 cannot be reached from X_7; we put a $\times$ in the X_1 position of the $\hat{\Gamma}\{X_7\}$ column. Next, we argue in exactly the same way in order to determine $\hat{\Gamma}^-\{X_7\}$ but the words 'row' and 'column' are interchanged, and in order to

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	$\hat{\Gamma}\{X_7\}$
X_1		1	1						×
X_2		1	1	1					2
X_3		1		1					1
X_4			1			1		1	2
X_5							1		1
X_6									1
X_7			1		1	1			0
X_8								1	3
$\hat{\Gamma}^-\{X_7\}$	×	×	×	×	1	×	0	×	

	X_1	X_2	X_3	X_4	X_6	X_8	$\hat{\Gamma}\{X_3\}$
X_1		1	1				×
X_2		1	1	1			1
X_3		1		1			0
X_4			1		1	1	1
X_6							2
X_8						1	2
$\hat{\Gamma}^-\{X_3\}$	1	1	0	1	×	×	

	X_1	X_6	X_8	$\hat{\Gamma}\{X_8\}$
X_1				×
X_6				×
X_8			1	0
$\hat{\Gamma}^-\{X_8\}$	×	×	0	

$$\begin{aligned}\mathscr{C}\{X_7\} &= \hat{\Gamma}\{X_7\} \cap \hat{\Gamma}^-\{X_7\} \\ &= \{X_2, X_3, X_4, X_5, X_6, X_7, X_8\} \\ &\quad \cap \{X_5, X_7\} = \{X_5, X_7\}\end{aligned}$$

$$\begin{aligned}\mathscr{C}\{X_3\} &= \hat{\Gamma}\{X_3\} \cap \hat{\Gamma}^-\{X_3\} \\ &= \{X_2, X_3, X_4, X_6, X_8\} \\ &\quad \cap \{X_1, X_2, X_3, X_4\} \\ &= \{X_2, X_3, X_4\}.\end{aligned}$$

$$\begin{aligned}\mathscr{C}\{X_8\} &= \hat{\Gamma}\{X_8\} \cap \hat{\Gamma}^-\{X_8\} \\ &= \{X_8\} \cap \{X_8\} \\ &= \{X_8\}.\end{aligned}$$

define a path we take the arrows reversed. We obtain finally the class of X_7 as $\{X_5, X_7\}$.

We remove the rows and columns X_5 and X_7 from the matrix; we now begin again on this sub-matrix. We have chosen the vertex X_3 arbitrarily as the one whose class we are looking for. And so on.

Having numbered the classes arbitrarily, we have:

$$\mathbf{A}_1 = \{X_5, X_7\}, \qquad \mathbf{A}_2 = \{X_2, X_3, X_4\},$$
$$\mathbf{A}_3 = \{X_8\}, \qquad \mathbf{A}_4 = \{X_1\}, \qquad \mathbf{A}_5 = \{X_6\}.$$

There remains only to trace the arcs which join the vertices of the classes (figure 37). Figure 38 shows the graph representing the existing connections between the classes; we see that this graph necessarily has no circuit (otherwise, where a circuit exists, the vertices belong to the same class).

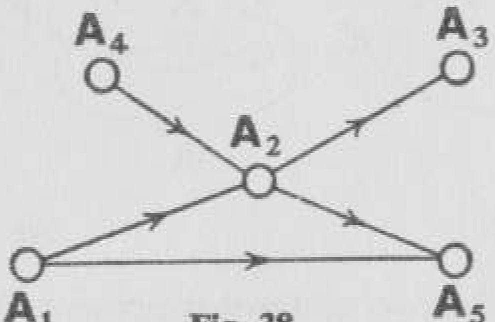


Fig. 38

It follows that the classes forming the strongly connected maximal sub-graphs of a graph have an ordered structure (partial or total according to the particular case) for the relation: 'there is a path from class $\mathbf{A}_i$ to the class $\mathbf{A}_j$',

Let us see what the concept just developed can do.

Consider a system (physical, biological, economic, sociological, psycho-sociological, etc.) able to take on a number of finite states†, and suppose that the possibilities of changes of

†The reasoning can be extended to systems having a non-finite denumerable number of states.

state are represented by a graph. Thus, given n states E_i, $i = 1, 2, \ldots, n$, the possibility of passing from state E_i to state E_j is represented by an arc (E_i, E_j). Suppose further that the possible changes of state of the system take place at times $t_0, t_1, t_2, \ldots$. The decomposition of the graph of the system into strongly connected maximal sub-graphs will give a necessary equilibrium condition for the system.

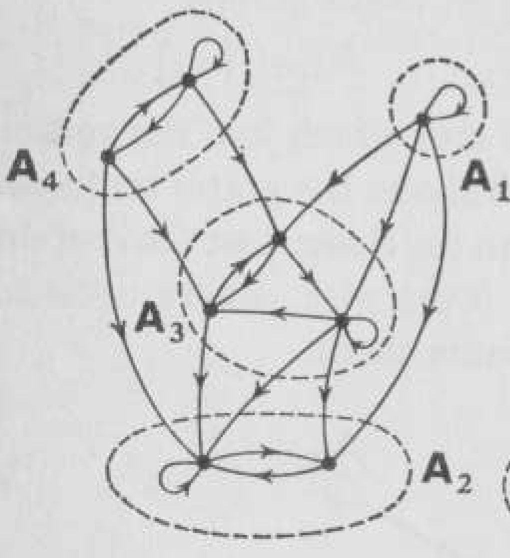


Fig. 39

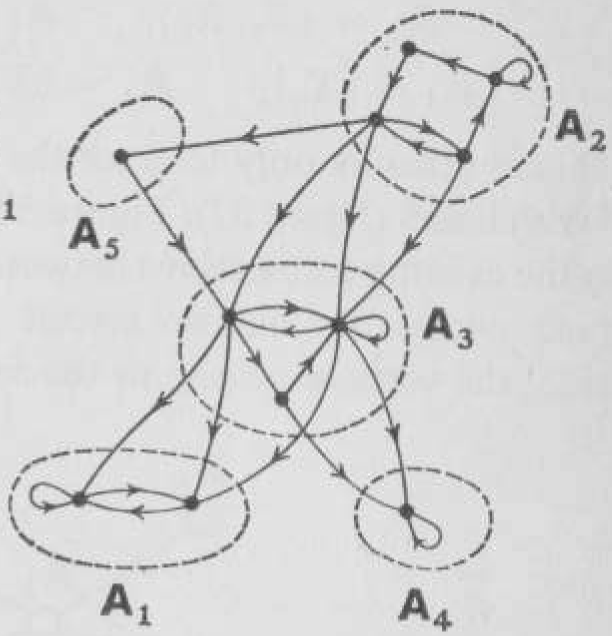


Fig. 40

Let us examine the two systems represented by figures 39 and 40. The former contains a class $\mathbf{A}_2$ such that, whenever the system is in a state of this class, it cannot leave $\mathbf{A}_2$, and such a class is unique. This is not the case with the latter, where there are two classes, $\mathbf{A}_1$ and $\mathbf{A}_4$, having this property. If we give a 'transition probability' to each arc of each of the graphs considered above, we obtain, under certain conditions, a *Markov chain*, a particularly important concept in genetics, in group communications, in economics, to cite only a few important applications. A Markov chain corresponding to the graph of figure 39 will possess a subset of states, such as $\mathbf{A}_2$, which will sooner or later cause the system to function in the

states of $\mathbf{A}_2$ only, this being perfectly predictable in advance; we say that the system is *ergodic*. This will not be the case for the Markov chain for which the graph is that of figure 40. For this one, we cannot predict in advance which of the subsets of states, $\mathbf{A}_1$ or $\mathbf{A}_4$, will be sooner or later the final subset; this system is not ergodic.

Let us look at some further explanations and an example of a Markov chain. Consider a system where we produce changes of state $E_i \to E_j$. Suppose we know the probabilities p_{ij} of the system being in the state E_j at time $t + 1$ after being in the state E_i at time t. If the p_{ij} are independent and if the initial probabilities $p_i(0)$ of the system being in state E_i at time $t = 0$ are given, we have completely defined a Markov chain. We then have:

$$(20) \quad p_i(t+1) = \sum_{j=1}^{n} p_{ij} \cdot p_j(t) \qquad \begin{array}{l} i = 1, 2, \ldots, n \\ t = 0, 1, 2, 3, \ldots \end{array}$$

where n is the number of states.

Using matrix notation, we can write equation (20) as:

$$(21) \qquad [p(t+1)] = [p(t)]\,[\mathscr{M}],$$

where $[p(t)]$ represents the vector with elements $p_i(t)$ and $[\mathscr{M}]$ represents the square matrix with elements p_{ij}. From (21), by putting t successively equal to 0, 1, 2, 3, . . . , we obtain:

$$(22) \quad [p(t+1)] = [p(0)].[\mathscr{M}]^{t+1}, \qquad t = 0, 1, 2, 3, \ldots.$$

This allows the calculation of the probabilities $p_i(t)$ for each value of t.

The matrix $[\mathscr{M}]$ with elements p_{ij} is such that:

$$(23) \qquad \sum_{j=1}^{n} p_{ij} = 1, \qquad i = 1, 2, \ldots, n;$$

$$p_{ij} \geqslant 0, \qquad i, j = 1, 2, \ldots, n$$

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
E_1	0.1	0	0.3	0	0.3	0	0	0	0.1	0.2
E_2	0	0	0	0	1	0	0	0	0	0
E_3	0	0	0	0	0	0	1	0	0	0
E_4	0.4	0	0	0.3	0	0	0	0.1	0.2	0
E_5	0	0.6	0	0	0	0.4	0	0	0	0
E_6	0	0	0	0	0	1	0	0	0	0
E_7	0	0	0	0	0	0	0	0	0	1
E_8	0.1	0	0	0.7	0	0	0	0.1	0.1	0
E_9	0.2	0.2	0	0	0.3	0	0	0	0.3	0
E_{10}	0	0	1	0	0	0	0	0	0	0

Fig. 41

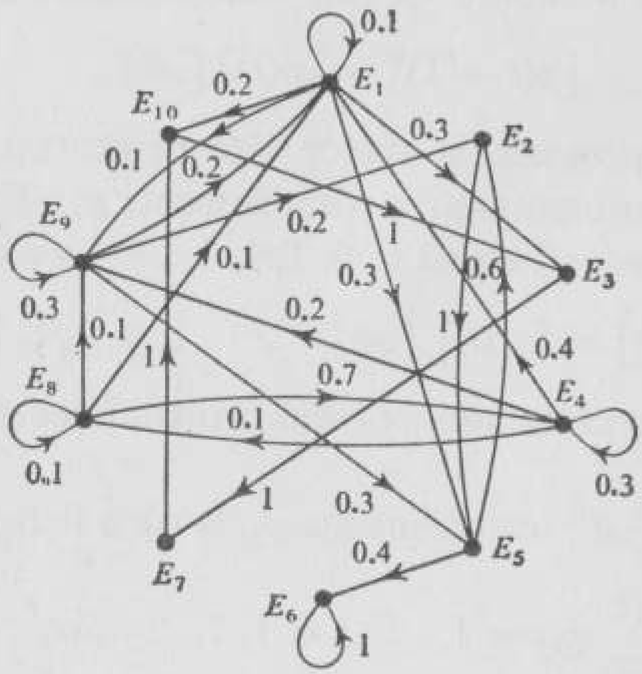


Fig. 42

We call it a *stochastic matrix*. All the elements are non-negative and the sum of the elements of the same row is equal to 1.

In the theory of Markov chains, we are concerned with what $[\mathcal{M}]^t$ becomes when $t \to \infty$. If all the rows of

$$[\tilde{\mathcal{M}}] = \lim_{t \to \infty} [\mathcal{M}]^t$$

are identical, we say that the matrix is *completely ergodic*, in which case there is a permanent condition for the Markov chain which stays in the states of the final class, which, we show, is unique in this case.†

Take, for example, the Markov chain given by figure 41 and suppose that the initial states have known non-zero probabilities. We can associate the graph of figure 42 with the matrix of figure 41, and its decomposition into strongly-connected maximal sub-graphs gives figure 43. We see that the matrix of figure 41 is not completely ergodic.

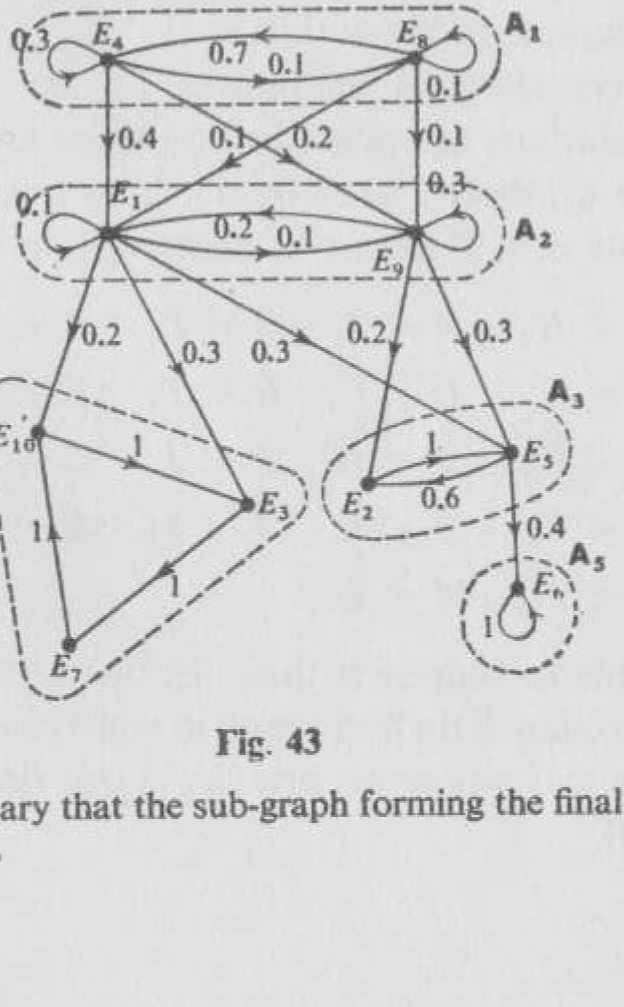


Fig. 43

†It is also necessary that the sub-graph forming the final class does not have cyclic subsets.

The modern theory of systems or 'general theory of automata' uses several concepts of the theory of graphs. The concept of *automata* considered here is wider than that of machines for handling information and languages, and it covers all that can be described by taking account of states in space and time, abstract or concrete. It is on account of the importance of this concept that we have from the start quoted the applications mentioned.

Ordinal Function of a Circuitless Graph

To introduce this concept, we are going to use a concrete example.

Consider a set of technological operations whose nature is such that certain of them are carried out before one or more others. We propose to put them in order so that all the relations of precedence are satisfied. This problem, well known in industrial activities, is easily resolved by the theory of graphs. We use an example to show this.

Take 13 technological operations, A, B, $\ldots$, M, whose precedence relations are given below; there are 27 of these in number. The symbol $\prec$ used in the table means 'necessarily precedes', thus $C \prec F$ means 'C necessarily precedes F'.

$$
\begin{array}{llllll}
A \prec G, & A \prec H, & A \prec J, & A \prec L, & B \prec A, & C \prec A, \\
C \prec B, & C \prec F, & D \prec C, & D \prec F, & E \prec B, & E \prec C, \\
E \prec D, & E \prec F, & F \prec H, & F \prec J, & G \prec L, & H \prec L, \\
H \prec M, & J \prec K, & J \prec L, & J \prec M, & K \prec I, & K \prec M, \\
L \prec I, & L \prec M, & M \prec I. & & &
\end{array}
$$

It is possible to represent these 13 operations and 27 constraints controlling them by a graph in which a vertex represents an operation and where an arc (X_i, X_j) is drawn if $X_i \prec X_j$ (see figure 44).

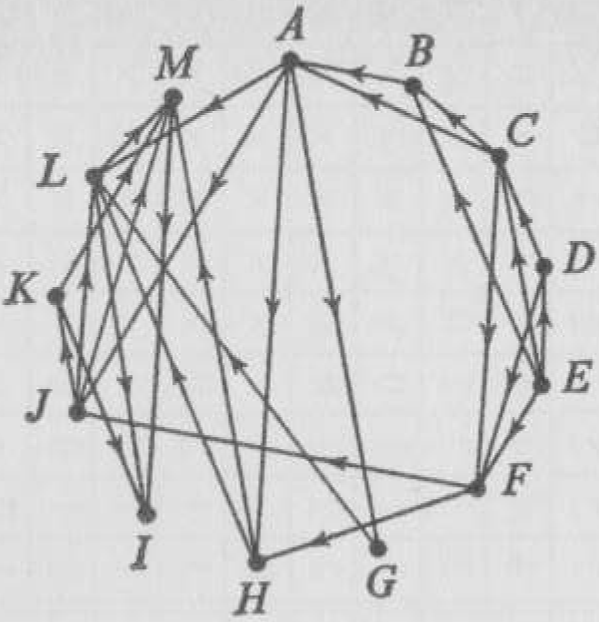


Fig. 44

It is advisable to note that this graph should not contain a circuit, otherwise an actual operation will be able to be prior to itself, which we reject in the light of the nature of the problem presented above.

Consider now the matrix of figure 45, which represents the same graph. Next to this matrix we have put a certain number of columns obtained in the way indicated later; furthermore, we have deliberately omitted to put 0 in the positions in the matrix where, by definition, it should be put, in order to simplify the writing down (see figure 45).

Denote the vectors representing the columns of this matrix by $V_A, V_B, \ldots, V_M$. Then calculate $V_0 = V_A + V_B + \ldots + V_M$; the result is shown as column V_0. This vector has a zero corresponding to row I; this means that the vertex (that is to say, the operation) is not followed by another. We say that vertex I is of degree 0. Now calculate $V_1 = V_0 - V_I$, and put a $\times$ in the Ith row of the vector V_1. There now appears in V_1 a new zero corresponding to the Mth row; therefore, if I is deleted then M will have no successor; we say that M is of

	A	B	C	D	E	F	G	H	I	J	K	L	M		V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
A							1	1		1		1			4	4	4	3	0	×	×	×	×
B	1														1	1	1	1	1	0	×	×	×
C	1	1				1									3	3	3	3	3	1	0	×	×
D			1			1									2	2	2	2	2	1	1	0	×
E		1	1	1		1									4	4	4	4	4	3	2	1	0
F								1		1					2	2	2	2	0	×	×	×	×
G												1			1	1	1	0	×	×	×	×	×
H												1	1		2	2	1	0	×	×	×	×	×
I															0	×	×	×	×	×	×	×	×
J											1	1	1		3	3	2	0	×	×	×	×	×
K									1				1		2	1	0	×	×	×	×	×	×
L									1				1		2	1	0	×	×	×	×	×	×
M									1						1	0	×	×	×	×	×	×	×
															I	M	K L	G H J	A F	B	C	D	E
															0	1	2	3	4	5	6	7	8

Fig. 45

degree 1. Calculate now $V_2 = V_1 - V_M$, and put a $\times$ in the vector V_2 where zeros appear in the previous vectors; two new zeros, corresponding to the Kth and Lth rows, appear in V_2. Thus, I and M being removed, K and L are not followed by another vertex; we say that K and L are of degree 2. Calculate $V_3 = V_2 - V_K - V_L$; three new zeros appear, ... and so on. Finally, we have classified the 13 vertices into 9 degrees: $N_0, N_1, N_2, \ldots, N_8$. These degrees define what we call the *ordinal function of a circuitless graph.*

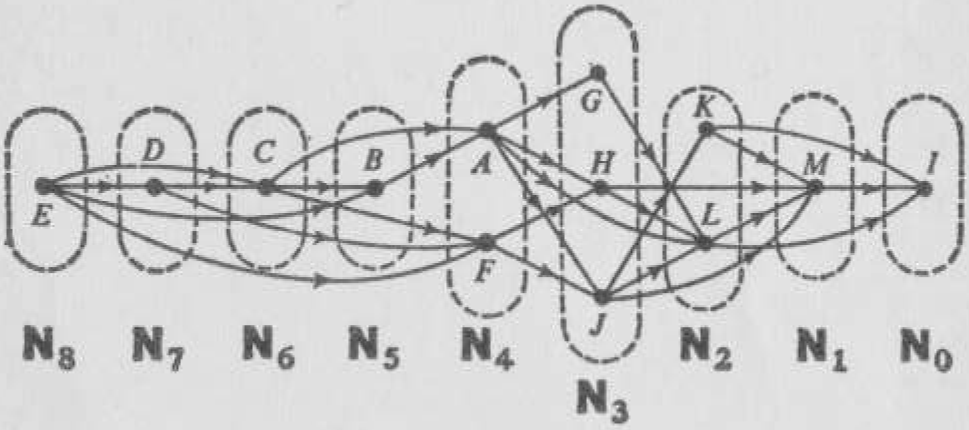


Fig. 46

Figure 46 gives a representation of this decomposition by degree where the numbering can be made from I or, in the other direction, from E. Thus, we can see in figure 46 the order in which the operations may be arranged, not only in relation to each other but also in the totality of the structure. We see that the operation I is a final operation, preceded by M, itself preceded by K and L (putting them in order arbitrarily), these being preceded by G, H and J (putting them in order arbitrarily), and so on.

Note that we will be able to look for another ordinal function arising from the inverse relation 'X_i follows X_j'. We take then the row vectors of the matrix and the ordinal function obtained may well be different, but there will always

be a total ordering of all the vertices compatible with each ordinal function.

The study of degrees, as considered above, can lead to important clarifications and to more efficient methods in scheduling and sequencing, in launching, in technological decisions relating to product ranges, in handling PERT diagrams (critical path method), and so on.

4. Elements having a Specified Property

We are now going to examine another category of problems, that of looking for a subset of permutations compatible with constraints.

Consider the following combinatorial problem: we are given six liquids, A, B, C, D, E and F, which can pass down a pipeline one after the other, but, on account of the nature of these liquids, certain products cannot follow in the pipeline. We use the following symbol: $X_i \dashv X_j$, 'X_i may not immediately precede X_j'.

Suppose that we have the constraints

$$A \dashv B,\quad A \dashv D,\quad A \dashv E,\quad A \dashv F,\quad B \dashv D,\quad B \dashv F,\quad C \dashv A,$$
$$C \dashv D,\quad C \dashv E,\quad D \dashv A,\quad D \dashv B,\quad D \dashv E,\quad E \dashv A,\quad E \dashv B,$$
$$E \dashv C,\quad F \dashv B,\quad F \dashv C,\quad F \dashv D.$$

We then construct a graph (figure 47) where an impossibility is represented by an arc going from A to B if $A \dashv B$, this being done for the 18 impossibilities.

This problem amounts to finding what are the permutations of the six products satisfying these constraints. We now produce the complementary graph $G^* = (\mathbf{E}, \mathbf{U}^*)$ (figure 48) of the graph $G = (\mathbf{E}, \mathbf{U})$ (figure 47), but omitting any loops.

Each permutation of the six elements satisfying the constraints gives us an elementary path of length 5 in G^*. An

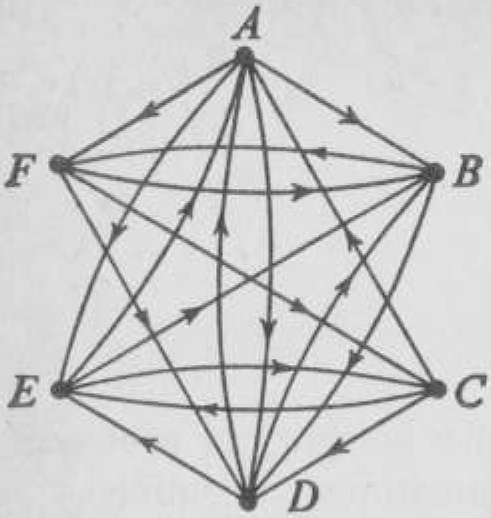


Fig. 47

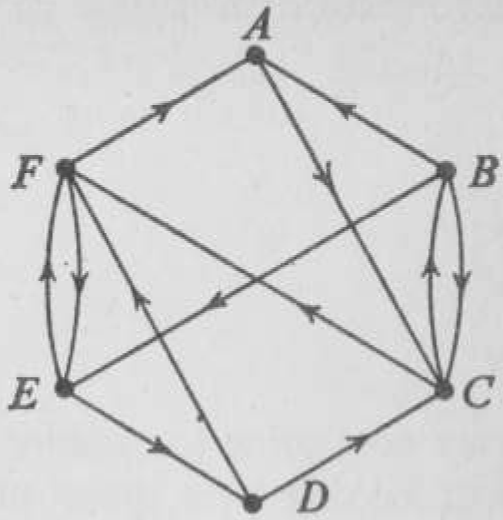


Fig. 48

elementary path of length $n-1$ in a graph with n vertices is called a *Hamiltonian path*. The problem considered here is thus that of determining the Hamiltonian paths of G^*.

We are going to use an enumeration procedure, without redundancy or omission, called *latin multiplication* or *concatenation.*

We shall call a series of vertices, forming a path and having a given property $\mathscr{P}$, a *latin sequence*. We shall define a certain operation on these sequences in the following way where we first decide to designate the empty set in the set of sequences by $\emptyset$. We shall also state for every latin sequence s

$$s * \emptyset = \emptyset,$$

$$\emptyset * s = \emptyset,$$

$$\emptyset * \emptyset = \emptyset, \tag{24}$$

Given two latin sequences possessing a property $\mathscr{P}$

$$s_1 = (X_{i_1}, X_{i_2}, \ldots, X_{i_{m-1}}, X_{i_m})$$

and

$$s_2 = (X_{j_1}, X_{j_2}, \ldots, X_{j_{n-1}}, X_{j_n}).$$

we state:

$$s_1 * s_2 = (X_{i_1}, X_{i_2}, \ldots, X_{i_{m-1}}, X_{i_m}, X_{j_2}, \ldots, X_{j_{n-1}}, X_{j_n})$$

(25) if $X_{i_m} = X_{j_1}$ and if the sequence obtained possesses the property $\mathscr{P}$,

$$s_1 * s_2 = \varnothing \text{ otherwise.}$$

Consider, for example, the graph of figure 48 and let us select as the property $\mathscr{P}$: 'the latin sequence is an elementary path of the graph'. Let us see how to use the operation we have defined in the example of figure 48.

Given:

$$s_1 = (F, E, D), \qquad s_2 = (D, C, B),$$

$$s_3 = (D, C, B, E), \qquad s_4 = (C, B, A).$$

These sequences have the property $\mathscr{P}$ (they form elementary paths). On applying the preceding rules, we have:

$$s_1 * s_2 = (F, E, D, C, B),$$

$s_1 * s_3 = \varnothing$ (since the sequence obtained has not got the property $\mathscr{P}$, E will appear twice in the sequence).

$s_1 * s_4 = \varnothing$ (since $D \neq C$).

In order to effect the calculation, without redundancy or omission, of the elementary paths, we will use a particular kind of matrix representation of the graph called the *latin matrix* associated with the graph (see figure 49). (To simplify things, the positions in the matrix in which there should appear the symbol $\varnothing$, because of the absence of an arc, will be left blank.) We will denote this matrix by $[M^{(1)}]$. Thus, at the intersection of row B and column C, we have BC because there is a path of length 1 (that is to say, an arc) from B to C. In the position cor-

responding to row *C* and column *E* we have left a blank space since there is no arc *CE*.

Next, call the same matrix, having all the initial vertices removed, $[M'^{(1)}]$. Thus, at the intersection of row *B* and column *C*, we have *C* instead of *BC* (see figure 50).

	A	B	C	D	E	F
A			AC			
B	BA		BC		BE	
C		CB				CF
D			DC			DF
E				ED		EF
F	FA				FE	

matrix $[M^{(1)}]$

Fig. 49

	A	B	C	D	E	F
A			C			
B	A		C		E	
C		B				F
D			C			F
E				D		F
F	A				E	

matrix $[M'^{(1)}]$

Fig. 50

Now, for $[M^{(1)}]$ and $[M'^{(1)}]$, let us define an operation $[M^{(1)}] * [M'^{(1)}]$ by putting in the matrix formed as a result of this operation the latin sequences having the property $\mathscr{P}$ (here, being an elementary path) at the intersection of row i and column j. These sequences will be obtained by considering the latin sequences contained in the ith row of the matrix $[M^{(1)}]$ together with those contained in the jth column of the matrix $[M'^{(1)}]$, namely, in considering together the elements $(i, 1)$ with $(1, j)$, $(i, 2)$ with $(2, j)$, . . . , (i, n) with (n, j) if the matrix is of order n (n rows and n columns). We thus work in a way which is very similar to that used in numerical matrix analysis (linear algebra).

It is in this way that the matrix $[M^{(2)}]$ has been obtained (see figure 51). Now, by performing the operation $[M^{(2)}] * [M'^{(1)}]$

in the same way, we obtain $[M^{(3)}]$ (see figure 52), which gives all the elementary paths of length 3.

Continuing, we have: $[M^{(3)}] * [M'^{(1)}] = [M^{(4)}]$ (see figure 53), and then $[M^{(4)}] * [M'^{(1)}] = [M^{(5)}]$ (see figure 54).

On stopping there, we have obtained all the Hamiltonian paths of the graph, ten in number:

$$\begin{array}{ll}
(A, C, B, E, D, F), & (B, E, D, C, F, A), \\
(B, E, D, F, A, C), & (B, A, C, F, E, D), \\
(C, B, E, D, F, A), & (D, C, B, E, F, A), \\
(D, F, A, C, B, E), & (E, D, F, A, C, B), \\
(F, E, D, C, B, A), & (F, A, C, B, E, D).
\end{array}$$

Given the associative property of latin matrices, we note that we can write:

$$(26) \quad [M^{(i)}] * [M'^{(j)}] = [M^{(j)}] * [M'^{(i)}] = [M^{(i+j)}].$$

Thus, in our example, $[M^{(3)}] * [M'^{(2)}] = [M^{(5)}]$ (see figure 55).

So, various simplifications can be introduced.

The same procedure can be used for finding elementary circuits, for arrangements where repetition is allowed, and for various other concepts where such calculation is useful. In practice, one can leave out the matrix writing, which was presented here in order to make explanations easier, and make use of suitably arranged tables. Besides, if the constraints and/or the elements are very numerous we have to turn to electronic analysis. The concept of *cell* introduced by Pair[1] allows the reduction of the overcrowding of memory in calculations of this kind programmed on a computer.

Let us give a further practical example where the search for permutations satisfying certain constraints can be useful. Manufactured products are made from a single raw material

	A	B	C	D	E	F
A			AC			
B	BA		BC		BE	
C		CB				CF
D			DC			DF
E				ED		EF
F	FA				FE	

$[M^{(1)}]$

*

	A	B	C	D	E	F
A			C			
B	A		C		E	
C		B				F
D			C			F
E				D		F
F	A				E	

$[M'^{(1)}]$

=

	A	B	C	D	E	F
A		ACB				ACF
B			BAC	BED		BCF BEF
C	CBA CFA				CBE CFE	
D	DFA	DCB			DFE	DCF
E	EFA		EDC			EDF
F			FAC	FED		

$[M^{(2)}]$

Fig. 51

	A	B	C	D	E	F
A		ACB				ACF
B			BAC	BED		BCF BEF
C	CBA CFA				CBE CFE	
D	DFA	DCB			DFE	DCF
E	EFA		EDC			EDF
F			FAC	FED		

$[M^{(2)}]$

$*$

	A	B	C	D	E	F
A			C			
B	A		C		E	
C		B				F
D			C			F
E				D		F
F	A				E	

$[M'^{(1)}]$

$=$

	A	B	C	D	E	F
A					ACBE ACFE	
B	BCFA BEFA		BEDC		BCFE	BACF BEDF
C				CBED CFED		CBEF
D	DCBA DCFA		DFAC		DCFE DCBE	
E	EDFA	EDCB	EFAC			EDCF
F		FACB	FEDC			

$[M^{(3)}]$

Fig. 52

	A	B	C	D	E	F
A					ACBE ACFE	
B	BCFA BEFA		BEDC		BCFE	BACF BEDF
C				CBED CFED		CBEF
D	DCBA DCFA		DFAC		DCFE DCBE	
E	EDFA	EDCB	EFAC			EDCF
F		FACB	FEDC			

$[M^{(3)}]$

*

	A	B	C	D	E	F
A			C			
B	A		C		E	
C		B				F
D			C			F
E				D		F
F	A				E	

$[M^{(1)}]$

=

	A	B	C	D	E	F
A				ACBED ACFED		ACBEF
B	BEDFA		BEFAC	BCFED	BACFE	BEDCF
C	CBEFA					CBEDF
D		DFACB				DCBEF
E	EDCBA EDCFA	EFACB	EDFAC			
F		FEDCB			FACBE	

$[M^{(4)}]$

Fig. 53

	A	B	C	D	E	F
A				ACBED ACFED		ACBEF
B	BEDFA		BEFAC	BCFED	BACFE	BEDCF
C	CBEFA					CBEDF
D		DFACB				DCBEF
E	EDCBA EDCFA	EFACB	EDFAC			
F		FEDCB			FACBE	

$[M^{(4)}]$

$*$

	A	B	C	D	E	F
A			C			
B	A		C		E	
C		B				F
D			C			F
E				D		F
F	A				E	

$[M'^{(1)}]$

$=$

	A	B	C	D	E	F
A						ACBEDF
B	BEDCFA		BEDFAC	BACFED		
C	CBEDFA					
D	DCBEFA				DFACBE	
E		EDFACB				
F	FEDCBA			FACBED		

$[M^{(5)}]$

Fig. 54

	A	B	C	D	E	F
A					ACBE ACFE	
B	BCFA BEFA		BEDC		BCFE	BACF BEDF
C				CBED CFED		CBEF
D	DCBA DCFA		DFAC		DCFE DCBE	
E	EDFA	EDCB	EFAC			EDCF
F		FACB	FEDC			

$[M^{(3)}]$

$\overset{L}{*}$

	A	B	C	D	E	F
A		CB				CF
B			AC	ED		CF EF
C	BA FA				BE FE	
D	FA	CB			FE	CF
E	FA		DC			DF
F			AC	ED		

$[M'^{(2)}]$

=

	A	B	C	D	E	F
A						ACBEDF
B	BEDCFA		BEDFAC	BACFED		
C	CBEDFA					
D	DCBEFA				DFACBE	
E		EDFACB				
F	FEDCBA			FACBED		

$[M^{(5)}]$

Fig. 55

but become subject to modifications, additions and various transformations, during the production process where it is appropriate, in going from a manufactured product $\mathscr{P}_i$ to a manufactured product $\mathscr{P}_j$, to modify the manufacturing process according to a corresponding cost c_{ij}. One can say that the running costs of the manufactured products virtually consist only of the costs c_{ij}, the other costs being negligible by comparison with these. The order in which the products are arranged in the manufacturing process are of great importance for the totality of manufacturing expenditure. On the other hand, various constraints, technological or other, can require a priori that certain changes $\mathscr{P}_i \to \mathscr{P}_j$ be set aside; the corresponding arcs will be removed from the graph. We are thus induced to look for suitable solutions which are formed by permutations of products satisfying the constraints. But, we will be concerned with looking for the permutation (or permutations) of minimal cost, out of which another kind of problem arises, which we shall discuss later (the 'travelling salesman problem').

REFERENCES

1. PAIR, C., *Etude de la notion de pile. Application à l'analyse synthaxique*. Thèse. Fac. Sciences, Nancy, 1966.

5. Looking for an Optimal Path

Let us begin by considering a very simple concrete example. In a certain town, we have in mind looking for what is the quickest route between two road junctions *D* and *G* (see figure 56). We suppose that we have determined the immediate durations between junctions where traffic is possible (there are one-way and two-way streets). In figure 57, we have indicated the quickest route (*D*, *H*, *C*, *G*), which takes 20 units of time. How can we calculate such a minimal path? We are going to demonstrate several methods.

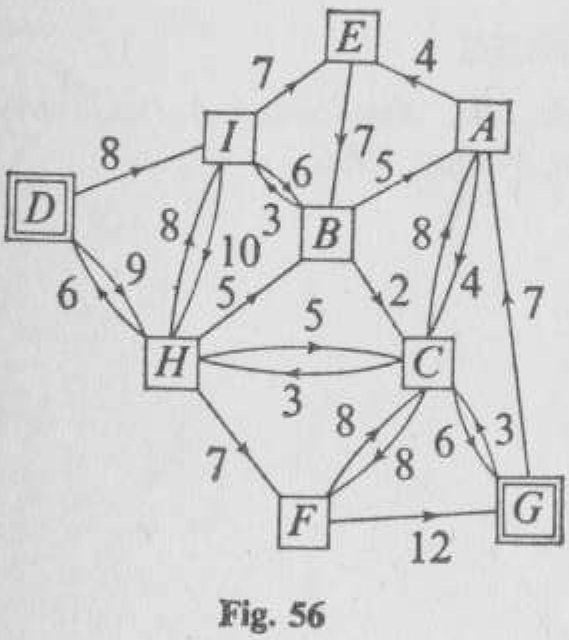


Fig. 56

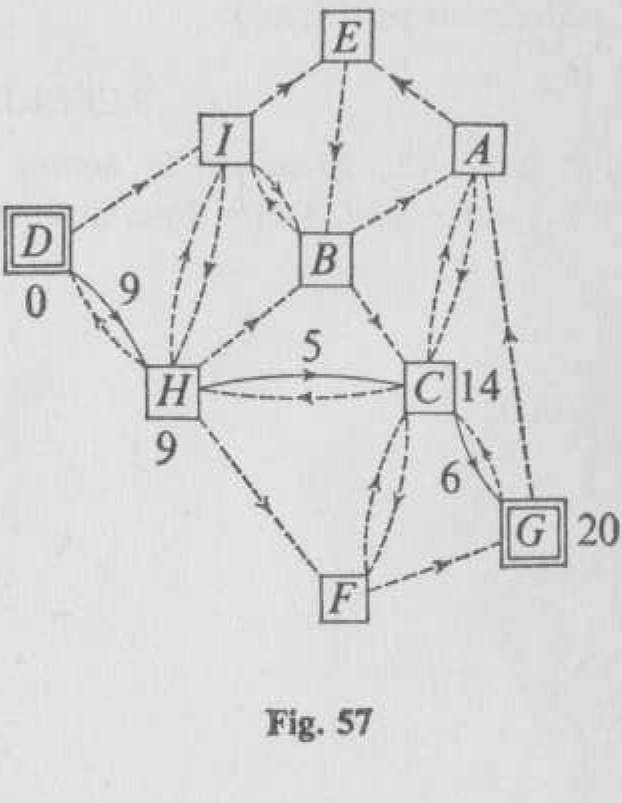


Fig. 57

Ford's Algorithm

Let X_0 be the original departure vertex from which we propose to calculate the path values to the other vertices. We put the value 0, then, against X_0 and values ∞ against the other vertices. The following rule is then applied.

If λ_i is the value against X_i, we look for an arc (X_i, X_j) such that $\lambda_j - \lambda_i > v(X_i, X_j)$, where $v(X_i, X_j)$ is the arc value; we then replace λ_j by $\lambda'_j = \lambda_i + v(X_i, X_j) < \lambda_j$. We continue in this manner until there is no arc which will allow us to decrease the λ_j. We then have the minimal values of all the paths from X_0 to the other vertices.

We are going to apply this algorithm to the example of figure 56.

Initial Assigning of Vertex Values

We assign 0 to the vertex D and ∞ to the others (see figure 58).

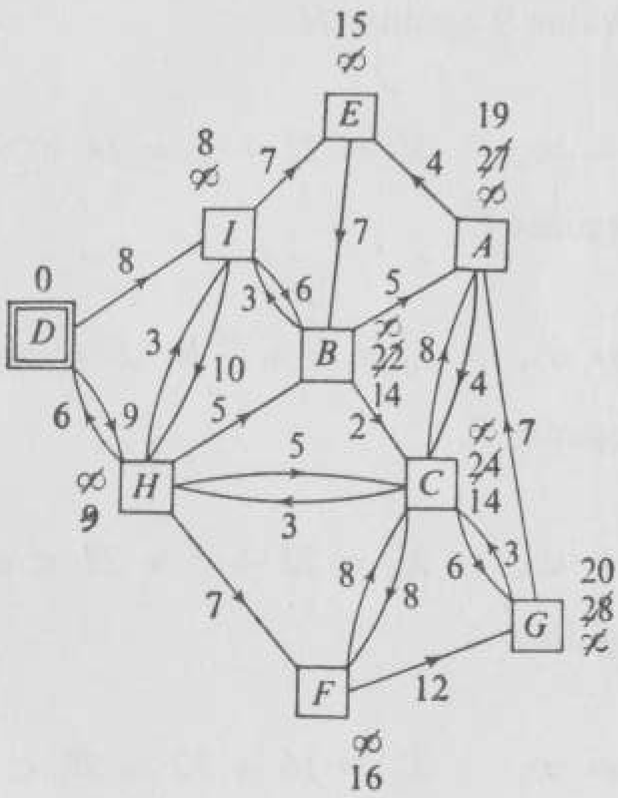


Fig. 58

Reduction of Vertex Values

We leave 0 against D.

We then have successively:

$$\lambda_I = \infty, \qquad \lambda'_I = 0 + 8 = 8 < \infty.$$

We give the value 8 to I, which replaces ∞ (on the figure we have struck out ∞ and put 8 in its place). Now, we have:

$$\lambda_E = \infty, \qquad \lambda'_E = 8 + 7 = 15 < \infty,$$

so we give E the value 15.

We have:

$$\lambda_B = \infty, \qquad \lambda'_B = 15 + 7 = 22 < \infty,$$

so we give B the value 22.

We have:

$$\lambda_H = \infty, \qquad \lambda'_H = 0 + 9 = 9 < \infty,$$

so we put the value 9 against H.

We have:

$$\lambda_C = \infty, \qquad \lambda'_C = 22 + 2 = 24 < \infty,$$

so we put 24 against C.

We have:

$$\lambda_F = \infty, \qquad \lambda'_F = 9 + 7 = 16 < \infty,$$

so we put 16 against F.

We have:

$$\lambda_A = \infty, \qquad \lambda'_A = 22 + 5 = 27 < \infty,$$

so we put 27 against A

We have:

$$\lambda_G = \infty, \qquad \lambda'_G = 16 + 12 = 28 < \infty,$$

so we put 28 against G.

We now start again with a new inspection of the vertices.

$$\lambda_I = 8, \qquad \lambda'_I = 0 + 8 = 8,$$

so we do not alter the value against I.

$$\lambda_H = 9, \qquad \lambda'_H = 8 + 10 = 18 > 9 \quad \text{or} \quad \lambda'_H = 0 + 9 = 9,$$

so we do not alter the value against H.

$$\lambda_E = 15, \qquad \lambda'_E = 8 + 7 = 15,$$

so we do not alter the value against E.

$$\lambda_B = 22, \qquad \lambda'_B = 15 + 7 = 22$$

or $$\lambda'_B = 8 + 6 = 14 < 22$$

or $$\lambda'_B = 9 + 5 = 14 < 22,$$

so we replace 22 against B by 14.

$$\lambda_C = 24, \qquad \lambda'_C = 14 + 2 = 16 < 24$$

or $$\lambda'_C = 9 + 5 = 14 < 24$$

or $$\lambda'_C = 16 + 8 = 24$$

or $$\lambda'_C = 28 + 3 = 31 > 24$$

or $$\lambda'_C = 27 + 4 = 31 > 24,$$

so we replace 24 against C by 14.

$$\lambda_F = 16, \qquad \lambda'_F = 9 + 7 = 16$$

or $$\lambda'_F = 14 + 8 = 22 > 16,$$

so we leave 16 against F.

$$\lambda_A = 27, \qquad \lambda'_A = 14 + 5 = 19 < 27$$

or $$\lambda'_A = 14 + 8 = 22 < 27$$

or $$\lambda'_A = 28 + 7 = 35 > 27,$$

so we replace 27 against A by 19.

$$\lambda_G = 28, \qquad \lambda'_G = 16 + 12 = 28$$

or

$$\lambda'_G = 14 + 6 = 20 < 28,$$

so we replace 28 by 20.

Continuing in this manner, we see that we cannot reduce any vertex value further. Figure 59 gives the optimal path values. Thus:

from D to I—(D, I): value 8,
D to H—(D, H): value 9,
D to E—(D, I, E): value 15,
D to B—(D, I, B) or (D, H, B): value 14,
D to C—(D, H, C): value 14,
D to F—(D, H, F): value 16,
D to A—(D, I, B, A) or (D, H, B, A): value 19,
D to G—(D, H, C, G): value 20.

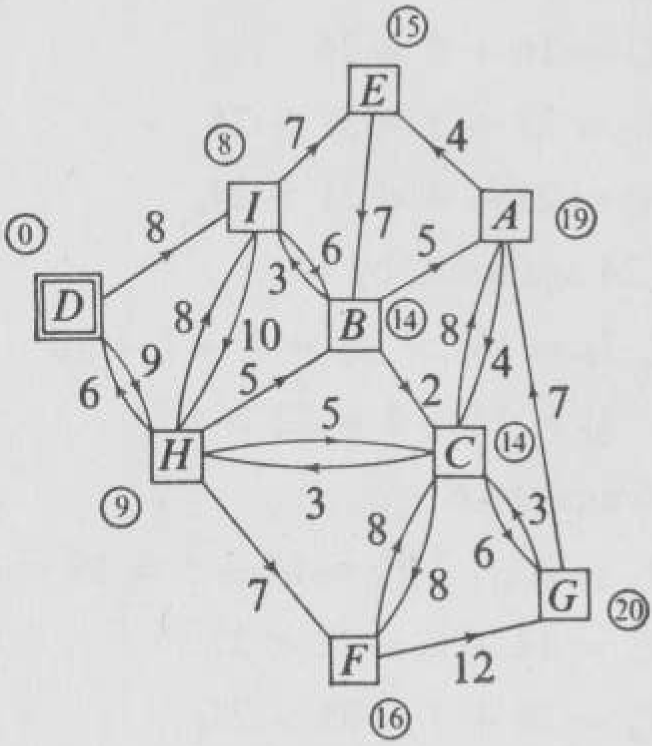


Fig. 59

We will go from junction D to each of the other junctions exclusively taking roads which allow us to give the minimal assigned value to each junction.

Use of Dynamic Programming

Another practical example will illustrate this usage. We are to construct a motorway between two towns, A and H, which will necessarily have access points in towns B, C, D, E, F and G (see figure 60). The possible sections have been investigated and for each of these we know the total cost: construction, design, and compulsory purchase. Let $x_0, x_1, x_2, \ldots, x_7$ be the variables representing the chosen access points. Thus

$x_0 \in \{A_1\}$, $x_1 \in \{B_1, B_2, B_3\}$, $x_2 \in \{C_1, C_2, C_3, C_4\}$, $x_3 \in \{D_1, D_2\}$, $x_4 \in \{E_1, E_2, E_3\}$, $x_5 \in \{F_1, F_2, F_3\}$, $x_6 \in \{G_1, G_2, G_3\}$, $x_7 \in \{H_1\}$.

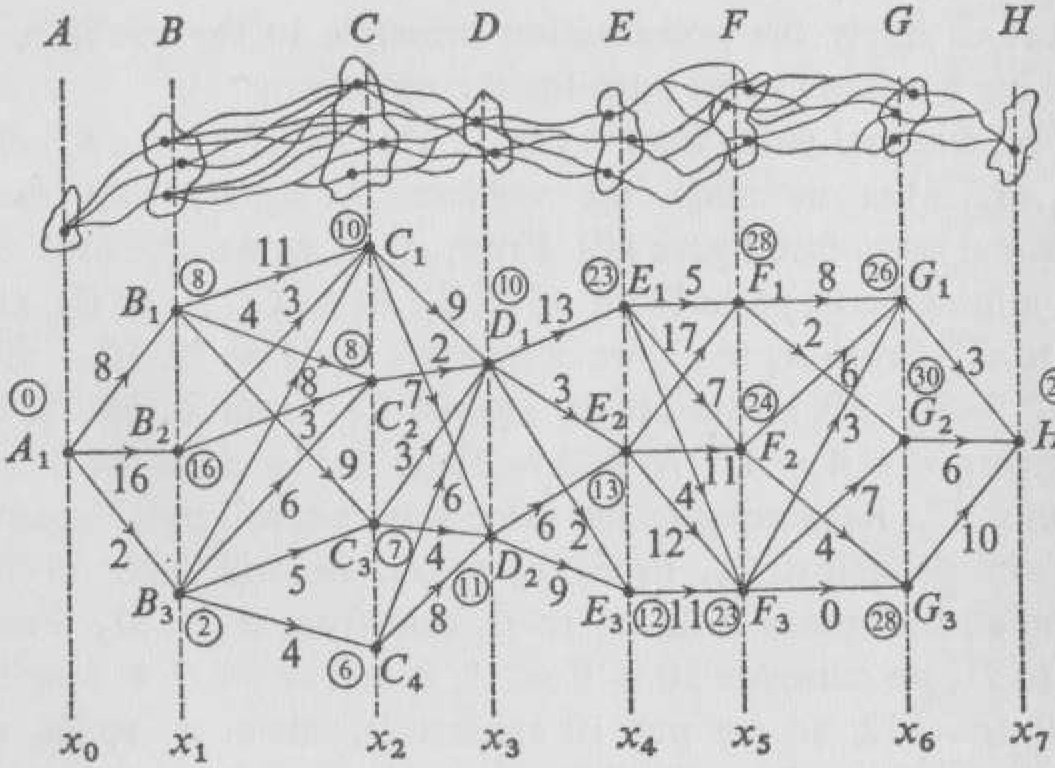


Fig. 60

The problem consists of finding a path, $x_0, x_1, x_2, x_2, x_4, x_5, x_6, x_7$, such that the total cost will be minimal.

We can use Ford's algorithm to calculate this, but the configuration of the graph allows us to use another method. We are going to use a famous principle, the *Bellman-Pontryagin optimisation principle*, which we now present in the form of a theorem:

An optimal path from vertex S_0 to vertex S_n can be formed only by sections of path $(S_i, \ldots, S_k, \ldots, S_j)$, where $0 \leqslant i, j \leqslant n$, which are optimal as between S_i and S_j.

The proof of this theorem is immediate. If there is a section of path from S_i to S_j which is better than $(S_i, \ldots, S_k, \ldots, S_j)$, for example $(S_i, \ldots, S_l, \ldots, S_j)$, the path $(S_0, \ldots, S_i, \ldots, S_l, \ldots, S_j, \ldots, S_n)$ will be better than $(S_0, \ldots, S_i, \ldots, S_k, \ldots, S_j, \ldots, S_n)$ and the latter will not be optimal, which is contrary to our hypothesis. We interpret, at will, according to the nature of the problem, the best word for 'maximal' or 'minimal'.

Let us apply the optimisation principle to the problem of seeking a minimal-cost path for the motorway.

The minimal paths from x_0 to x_1 are (A_1, B_1), (A_1, B_2) and (A_1, B_3). Let us mark the vertices B_1, B_2, B_3 with these minimal costs (see figure 60). From x_0 to x_2 we will look for the minimal-cost paths from A_1 to C_1, A_1 to C_2, A_1 to C_3, and A_1 to C_4. From A_1 to C_1 we compare $8 + 11 = 19$, $16 + 3 = 19$, $2 + 8 = 10$, so we put 10 against C_1; from A_1 to C_2, we compare $8 + 4 = 12$, $16 + 3 = 19$, $2 + 6 = 8$, so we put 8 against C_2. Reasoning in the same way, we will put 7 against C_3 and 6 against C_4. From x_0 to x_3, we will look for the minimal-cost paths from A_1 to D_1 and from A_1 to D_2. From A_1 to D_1 we compare $10 + 9 = 19$, $8 + 2 = 10$, $7 + 3 = 10$, $6 + 6 = 12$, so we put 10 against D_1; from A_1 to D_2 we compare $10 + 7 = 17$, $7 + 4 = 11$, $6 + 8 = 14$, so we

put 11 against D_2. And we continue up to x_7. We find that the minimal path has a cost-value 29. To find the actual minimal-value path (or paths), it is sufficient to go back; wherever the differences between circled values at the ends of arcs have the cost-value of the arc or section, the path goes along the arc.

This is how we obtain the two minimal-cost paths shown in figure 61.

Furthermore, we will be able to seek out this same optimum going from x_n to x_0. Moreover, the sense of the arrows on the arcs representing the sections is arbitrary provided that it is the same for each.

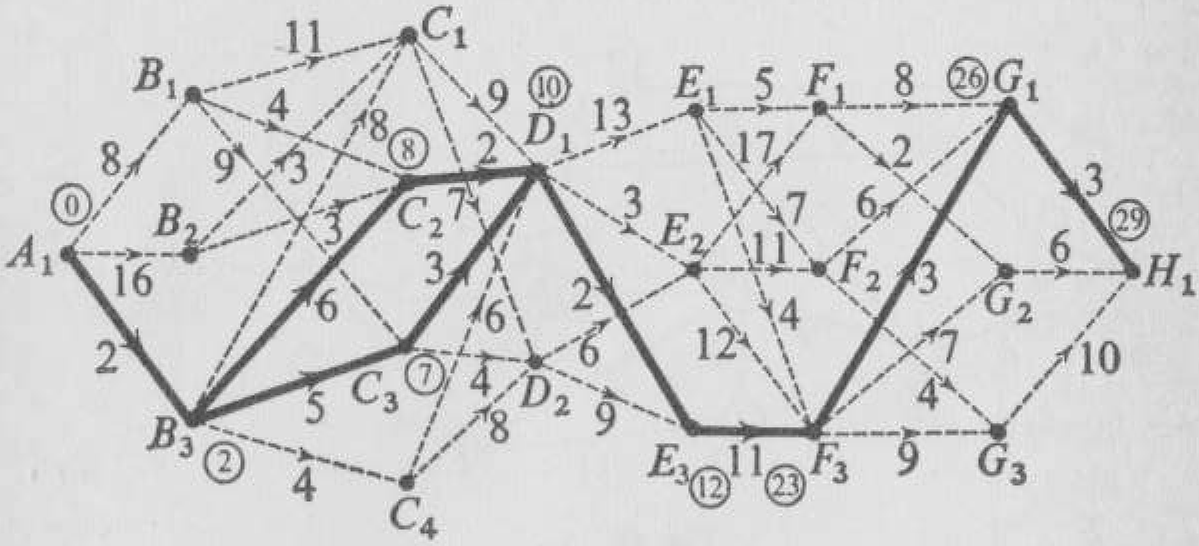


Fig. 61

PERT

PERT signifies Program Evaluation and Review Technique. This method consists of setting up a time chart of technological operations whose completion permits the definition of event-points forming the vertices of a graph, these vertices being connected together by the operations under consideration. We then seek to determine the earliest and latest times of completion of each event; from this information determined from the graph we can proceed to the adjustments, permitted according

to the chosen criterion, so as to reduce the durations or costs, acting on the parameters at our disposal.

Consider the example of figure 62. There are 20 operations allowing the definition of 12 events. The arcs of the graph are the operations, the vertices are the events. We have assigned a letter to each event, arranged so that the order of the letters accords with the ordinal function of the graph in starting with the initial event A; this makes the calculations easier but it is not essential. The optimisation process which we are going to use is that of dynamic programming, the order being that of looking for an optimal path; if we were to use Ford's algorithm, the order of letters would not matter.

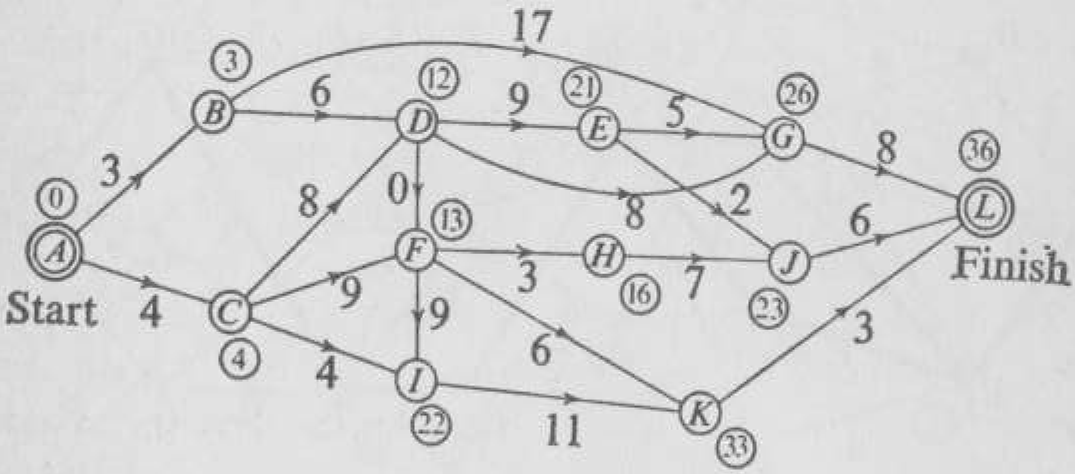


Fig. 62

We first look for the earliest time of completion of each event. Let us begin with the final event L. It is clear that this event cannot be completed *until all the operative paths from A to L have been completed*, and, in particular, until the longest path has been completed. Let us suppose that the durations of the operations are known (the numbers are placed by the arrows of the arcs). We are then brought back to the problem of finding the longest path or maximal-value path between A and L. Let us use dynamic programming in following the order of the letters. Against A, we put 0; against B, we put 3; against C, we put 4.

For D, we compare $3 + 6 = 9$ and $4 + 8 = 12$, and so we put 12. Against E, we put $12 + 9 = 21$. For F, we compare $12 + 0 = 12$ and $4 + 9 = 13$, so we put 13. For G, we compare $3 + 17 = 20$, $21 + 5 = 26$ and $12 + 8 = 20$, so we put 26. And so on, right up to L where we put 36. Thus the event L cannot take place until time 36. (We have not specified the unit of time chosen, this depends on the nature of the problem.) In order to find the corresponding path of maximal duration, we go back from L to A; the path goes through all the operations where the differences between the times put at the vertices equals the time put on the arcs. In this way, we have obtained the path shown in heavy hatching in figure 63. This path is called a *critical path* because all the events and operations which it includes cannot occur any later without delaying the time of the completion of the work. Thus, on the critical path the *earliest time* and the *latest time* are coincident. What is then the delay at our disposal for the other non-critical events? To obtain the latest times of these events, it suffices to determine, for each of them, the longest path in reversing the sense of the arrows, starting from L. We know what to do. To gain time, we will record the latest times in travelling back along the order of the letters. Thus, L must finish at 36, J must finish at the latest at $36 - 6 = 30$, H must finish at the latest at $30 - 7 = 23$, G must finish at $36 - 8 = 28$. For E, we must compare $28 - 5 = 23$ and $30 - 2 = 28$, we take the smaller number, that is 23. For D, we compare $23 - 9 = 14$ with $28 - 8 = 20$, and $13 - 0 = 13$, so we take 13, and so on. In figure 63, the earliest times are put in the squares to the left and the latest times in the squares to the right. Thus we know all the allowable delays for the non-critical events: 7 for J, 7 for H, 2 for G, 2 for E, 1 for D, and 4 for B. The works manager can control the working continuously from these essential figures.

A variation of the PERT method, where the vertices of the

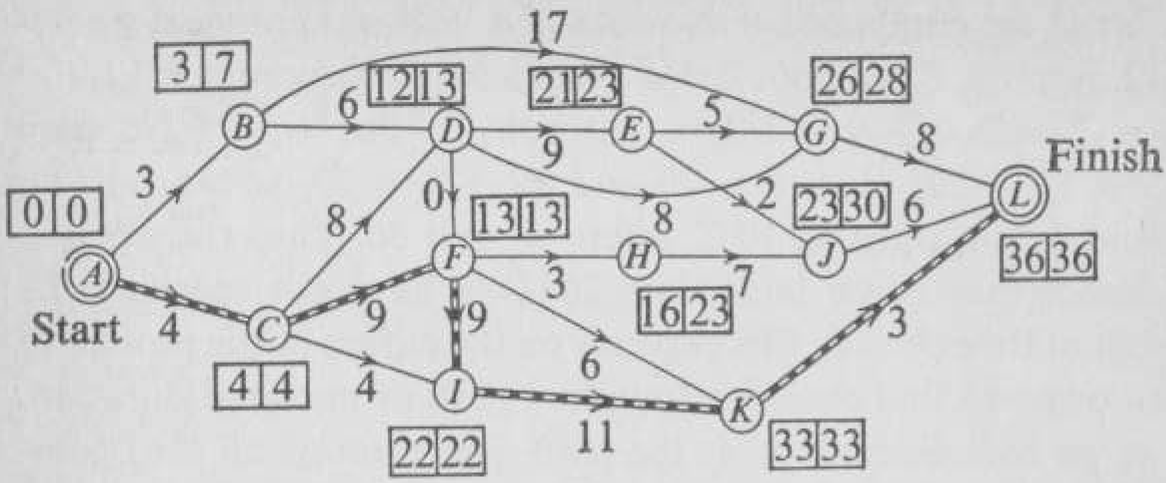


Fig. 63

graph are operations or tasks and the arcs are the time constraints between the start of one operation and that of the following operation, is called the *method of potentials.* The principle of seeking for and getting a critical path remains the same.

These methods, based on looking for the critical path, are used all over the world. (For more detail consult reference (1) given at the end of this chapter.)

Method of Separation and Progressive Evaluation
Branch and Bound Method

Let $\mathbf{E} = \{S_1, S_2, \ldots, S_n\}$ be a countable set of solutions of some problem. Let there be a function $f(S_i)$ for each solution and let us propose to determine the subset $\mathbf{E}_m$ of solutions corresponding to the minimum of $f(S_i)$. (In another problem, we will propose by contrast to determine the subset $\mathbf{E}_m$ of solutions corresponding to the maximum of $f(S_i)$.)

Let $\mathscr{P}_{\mathbf{A}}$ be a property permitting us to make a dichotomy of $\mathbf{E}$ into two complementary subsets $\mathbf{A}$ and $\overline{\mathbf{A}}$. Suppose that we can find a lower bound b_0 to the value of the solution elements of $\mathbf{E}$, then a lower bound $b_1 \geqslant b_0$ to the value of the solution elements of $\mathbf{A}$, and finally a lower bound $b'_1 \geqslant b_0$ to that of the

solution elements of $\overline{\mathbf{A}}$. Let us now consider another separating property $\mathscr{P}_{\mathbf{B}}$, the property $\mathscr{P}_{\mathbf{A}} \wedge \mathscr{P}_{\mathbf{B}}$ ($\mathscr{P}_{\mathbf{A}}$ and $\mathscr{P}_{\mathbf{B}}$) is then associated with $\mathbf{A} \cap \mathbf{B}$, similarly, $\mathscr{P}_{\mathbf{A}} \wedge \overline{\mathscr{P}}_{\mathbf{B}}$ with $\mathbf{A} \cap \overline{\mathbf{B}}$, $\overline{\mathscr{P}}_{\mathbf{A}} \wedge \mathscr{P}_{\mathbf{B}}$ with $\overline{\mathbf{A}} \cap \mathbf{B}$, and $\overline{\mathscr{P}}_{\mathbf{A}} \wedge \overline{\mathscr{P}}_{\mathbf{B}}$ with $\overline{\mathbf{A}} \cap \overline{\mathbf{B}}$ (see footnote †). Let us continue thus by introducing at each step another separating property, distinct from the preceding ones, $\mathscr{P}_{\mathbf{C}}, \mathscr{P}_{\mathbf{D}}, \ldots$, etc. We then construct a graph which we call a *directed tree*. Starting out from an original vertex or *root* of the directed tree, we can reach all the other vertices by means of a unique path, otherwise the graph is circuitless. (We will return to more details of this concept of directed tree later on.)

Let us then construct a directed tree (see figure 64) by giving ourselves as the rule of non-progression in successive dichotomies only the condition that all the lower bounds, evaluated at the vertices where we have previously ended, have a value greater than or equal to that of the vertex from which we propose to continue. If, in following this rule, we can end at a subset containing one and only one solution, this solution is minimal. This can easily be shown.

In essence, the intricate problem consists in the choice of separating properties and of the procedure which allows the step-by-step restriction of the separated subsets.

We are going to apply this method to a very well-known problem which returns to the search for a Hamiltonian circuit (or circuits) of minimal value in a graph.

The Travelling Salesman Problem

This algorithm is complicated to explain and to demonstrate. In this present popular work, we can give only an outline of the method, but the more interested reader will find all the

†We assume that the reader knows the symbols $\wedge$ and $\cap$ (logical product and intersection), and also $\vee$ and $\cup$ (logical sum and union). *See, e.g., Flegg*[3].

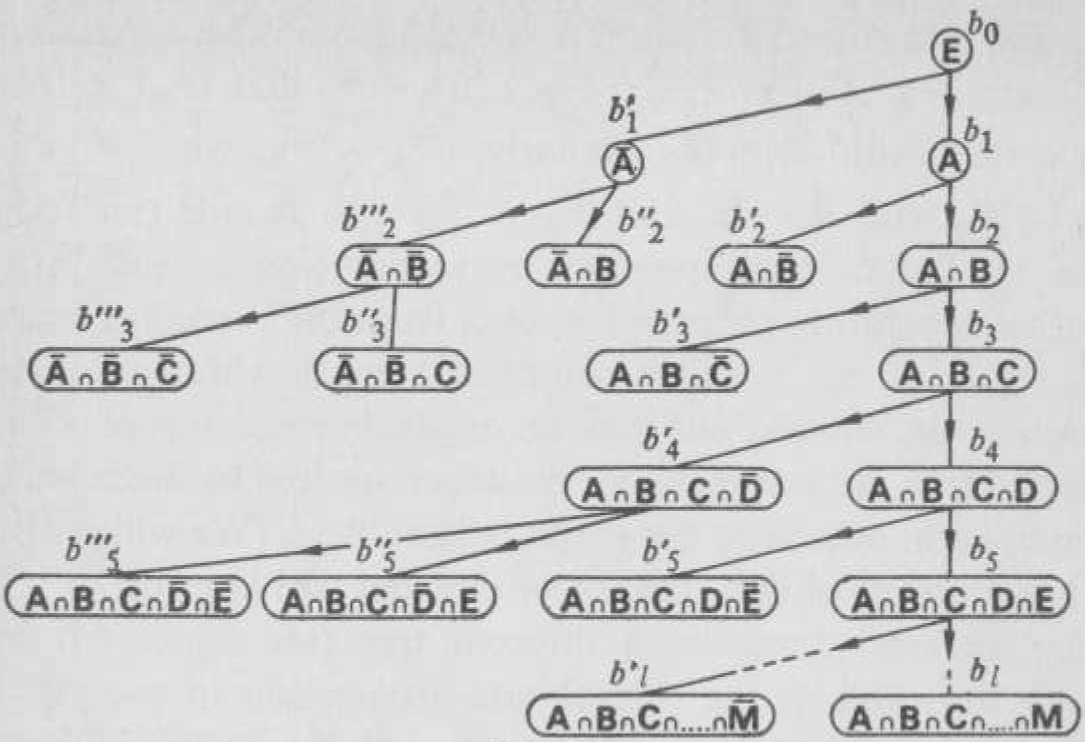


Fig. 64

rigorous explanations in Little's article[2]. The method is often referred to as *Little's algorithm.*

The problem known as the 'Travelling Salesman Problem' has long remained without a general analytic solution. In 1963, J. D. C. Little and several others presented a rigorous optimisation method applicable to this problem. The name of the problem comes from the fact that a Hamiltonian circuit is what a travelling salesman would follow in leaving a town X_i passing through all the other towns once and once only in order to return to X_i.

Let us consider an example, namely, that of the complete symmetric graph of figure 65. There is value $v(X_i, X_j)$, given by the table of figure 66, attached to each arc (X_i, X_j). Let us note that the method used would be equally well suited to any graph where we assume the existence of at least one Hamiltonian circuit. We have given a solution in figure 67, and the corresponding values are shown in figure 68.

To find a minimal-value Hamiltonian circuit we are first

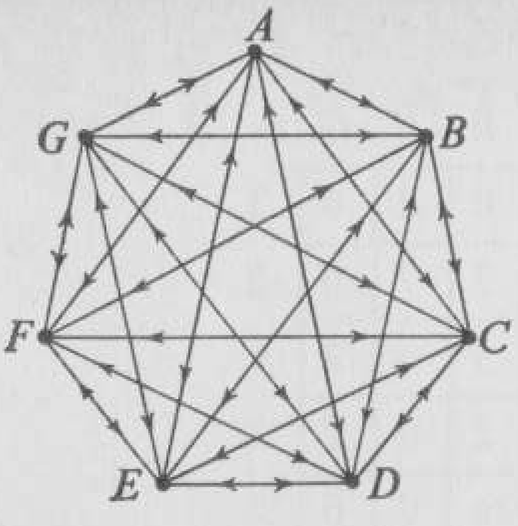


Fig. 65.

	A	B	C	D	E	F	G
A	∞	5	9	6	3	5	9
B	8	∞	8	8	5	9	2
C	6	9	∞	1	6	7	3
D	7	11	4	∞	4	2	9
E	4	6	3	2	∞	2	8
F	5	2	2	8	4	∞	3
G	8	1	3	16	5	3	∞

Fig. 66

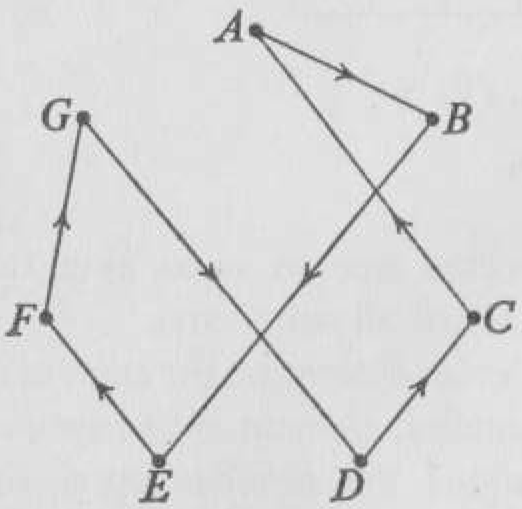


Fig. 67

	A	B	C	D	E	F	G
A		5					
B					5		
C	6						
D			4				
E						2	
F							3
G				16			

Total : 41

Fig. 68

going to transform the table of figure 66 so as to make a zero appear at least in each row and each column until we have exhausted the possibility of subtracting the same number from all the elements of a row or a column.

This is how in deducting 3 in row A, 2 in row B, . . . , and so on, then 2 in column A, we obtain the table given in figure 69. The sum of the numbers deducted being 15, we can state that

the set of solution values of the given problemhas a lower bound or is equal to 15.

	A	*B*	*C*	*D*	*E*	*F*	*G*	
A	∞	2	6	3	0	2	6	3
B	4	∞	6	6	3	7	0	2
C	3	8	∞	0	5	6	2	1
D	3	9	2	∞	2	0	7	2
E	0	4	1	0	∞	0	6	2
F	1	0	0	6	2	∞	1	2
G	5	0	2	15	4	2	∞	1
	2							

vertex ①

Fig. 69

In figure 70, we begin the directed tree so as to have the value 15 against its vertex **E** (the set of all solutions).

Now, in the table of figure 69, let us determine for each case where there is a 0 the sum of the smallest element in its row and the smallest element in its column; the number obtained, added to the bound already found (that is to say, 15), gives a new lower bound corresponding to the case of subsets of solu. tions not travelling through the arc corresponding to the cell- To have a highest possible number we examine all the sums thus found and take the largest.

The table shown in figure 69 has been reproduced in figure 71. For the case *AE*, we have $2 + 2 = 4$; for the case *BG*, we have $3 + 1 = 4$; for the case *CD*, we have $2 + 0 = 2$, and so on. Finally, the largest number so obtained is 4. With two cases corresponding to this 4, let us take *BG* arbitrarily. We can write

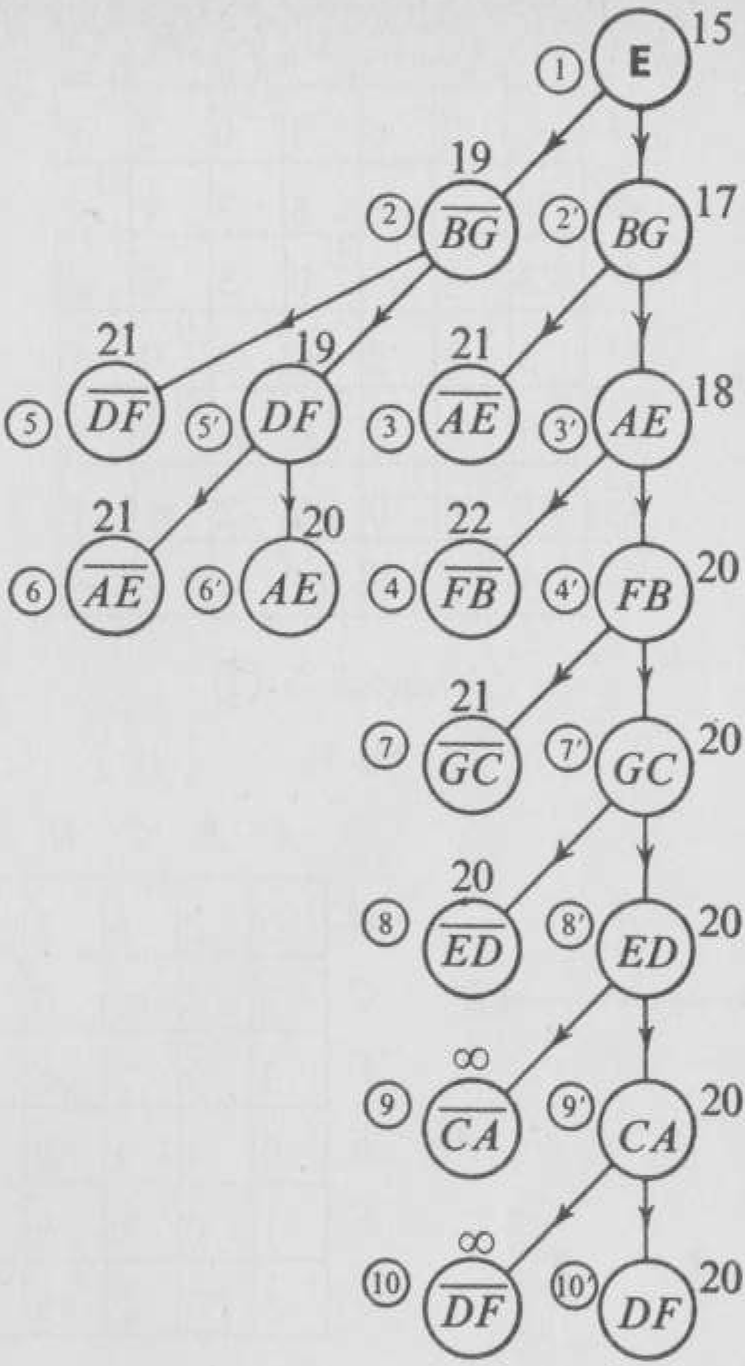


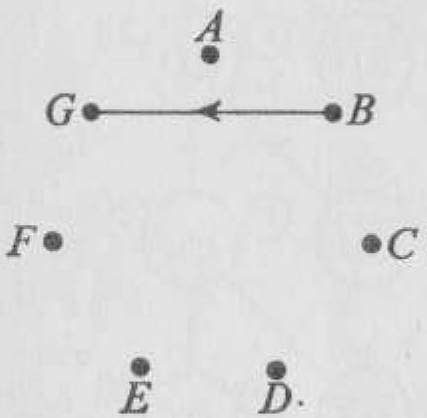
Fig. 70

down that the subset of solutions not travelling through *BG* has a lower bound 15 + 4 = 19. Let us put this 19 next to the vertex denoted by *BG* in the directed tree. To find a lower bound of the subset of solutions which, on the contrary, travel through *BG*, let us remove row *B* and column *G* from the table, thus obtaining figure 72.

	A	B	C	D	E	F	G
A	∞	2	6	3	$\boxed{4}$ 0	2	6
B	4	∞	6	6	3	7	$\boxed{4}$ 0
C	3	8	∞	$\boxed{2}$ 0	5	6	2
D	3	9	2	∞	2	$\boxed{2}$ 0	7
E	$\boxed{1}$ 0	4	1	$\boxed{0}$ 0	∞	$\boxed{0}$ 0	6
F	1	$\boxed{0}$ 0	$\boxed{1}$ 0	6	2	∞	1
G	5	$\boxed{2}$ 0	2	15	4	2	∞

vertex ②

Fig. 71



	A	B	C	D	E	F
A	∞	2	6	3	0	2
C	3	8	∞	0	5	6
D	3	9	2	∞	2	0
E	0	4	1	0	∞	0
F	1	0	0	6	2	∞
G	5	$\boxed{\infty}$	2	15	4	2

vertex ②

Fig. 72

Replace the number in the *GB* cell by ∞, otherwise the subset would be able to include solutions with circuits (G, B, G); such solutions are to be excluded. In figure 71 we see that it is possible to subtract 2 in row G, and we then arrive at figure 73

where we can assert that the subset of solutions travelling through *BG* is bounded below by 15 + 2 = 17. Let us put 17 against vertex *BG* of the directed tree (figure 70). Since 17 is less than 19, let us continue from vertex *BG* of the directed tree.

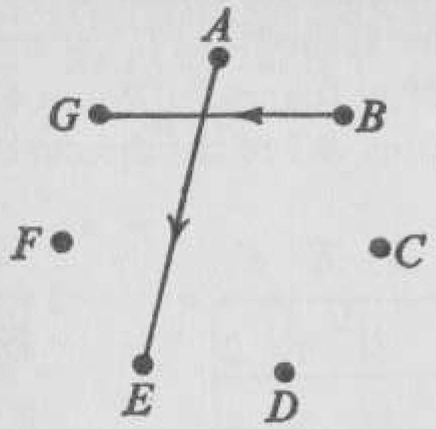
	A	*B*	*C*	*D*	*E*	*F*	
A	∞	2	6	3	$\boxed{4}$ 0	2	
C	3	8	∞	$\boxed{3}$ 0	5	6	
D	3	9	2	∞	2	$\boxed{2}$ 0	
E	$\boxed{1}$ 0	4	1	$\boxed{0}$ 0	∞	$\boxed{0}$ 0	
F	1	$\boxed{2}$ 0	$\boxed{0}$ 0	6	2	∞	
G	3	∞	$\boxed{0}$ 0	13	2	$\boxed{0}$ 0	2

vertex ③

Fig. 73

Let us then consider the subset of solutions travelling through *BG* but not through another arc which we are going to choose as we did with *BG*. On looking at the numbers obtained in the cells where there are 0's, we see that the cell *AE* gives the number 4, the largest such number. We choose *AE*, and we put 17 + 4 = 21 against the vertex *AE* of the directed tree. Let us then remove row *A* and column *E* of figure 73 to give us figure 74, where we put ∞ in the *EA* cell so as not to form a path which is not Hamiltonian. We can deduct 1 in column *A* so as to get figure 75. This 1 will be added to the preceding bound 17 to give 17 + 1 = 18, which is a lower bound of the subset of Hamiltonian circuits passing through the arcs *BG* and *AE*.

In the table of figure 75, the cells where there is a 0 give the number 4 for the greatest number which can increase the



	A	B	C	D	F
C	3	8	∞	0	6
D	3	9	2	∞	0
E	$\boxed{\infty}$	4	1	0	0
F	1	0	0	6	∞
G	3	∞	0	13	0

vertex 3'

Fig. 74

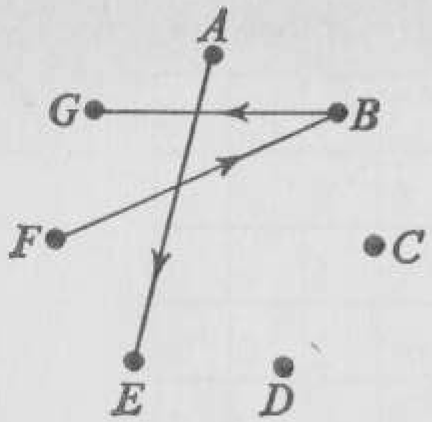
	A	B	C	D	F
C	2	8	∞	$\boxed{2}$ 0	6
D	2	9	2	∞	$\boxed{2}$ 0
E	∞	4	1	$\boxed{0}$ 0	$\boxed{0}$ 0
F	$\boxed{2}$ 0	$\boxed{4}$ 0	$\boxed{0}$ 0	6	∞
G	2	∞	$\boxed{0}$ 0	13	$\boxed{0}$ 0
	1				

vertex 4

Fig. 75

bound 18, and this corresponds to cell FB. The vertices FB and $\overline{FB}$ are now included in the directed tree; we put the bound $18 + 4 = 22$ against the vertex $\overline{FB}$.

Let us remove row F and column B and put ∞ in the GF cell so as not to form a circuit. We now go to figure 76 where it appears that we can subtract 2 in column A to get figure 77. Thus, the bound corresponding to vertex FB of the directed tree is: $18 + 2 = 20$.



	A	C	D	F
C	2	∞	0	6
D	2	2	∞	0
E	∞	1	0	0
G	2	0	13	$\boxed{\infty}$

vertex 4'

Fig. 76

	A	C	D	F
C	0	∞	0	6
D	0	2	∞	0
E	∞	1	0	0
G	0	0	13	∞
	2			

vertex 4'

Fig. 77

But 20 is greater than 19, so we must go back to the vertex $\overline{BG}$ of the directed tree. We again take the table of figure 69, in which we will put ∞ in the *BG* cell (the Hamiltonian circuit cannot pass through *BG*). We then remove 3 in row *B* and 1 in column *G*; we now get figure 78.

We now leave the reader to go from figure 79 to figure 90. The order in which the vertices are taken account of in the directed tree of figure 70 is indicated by the encircled ordering numbers. Finally, we arrive (see figures 89 and 90) at a Hamil-

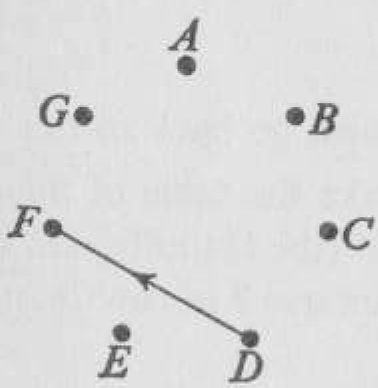
	A	B	C	D	E	F	G
A	∞	2	6	3	$\boxed{2}$ 0	2	5
B	1	∞	3	3	$\boxed{1}$ 0	4	∞
C	3	8	∞	$\boxed{1}$ 0	5	6	1
D	3	9	2	∞	2	$\boxed{2}$ 0	6
E	$\boxed{1}$ 0	4	1	$\boxed{0}$ 0	∞	$\boxed{0}$ 0	5
F	1	$\boxed{0}$ 0	$\boxed{1}$ 0	6	2	∞	$\boxed{1}$ 0
G	5	$\boxed{2}$ 0	2	15	4	2	∞

vertex ⑤

Fig. 78

tonian circuit of minimal value equal to 20. It is easy to verify that the minimal-value circuit is not unique. By continuing the directed tree with the vertices of bound 20, the reader can discover the other solutions with value 20.

The method of separation and progressive evaluation can



	A	B	C	D	E	G
A	∞	2	6	3	0	5
B	1	∞	3	3	0	∞
C	3	8	∞	0	5	1
E	0	4	1	0	∞	5
F	1	0	0	$\boxed{\infty}$	2	0
G	5	0	2	15	4	∞

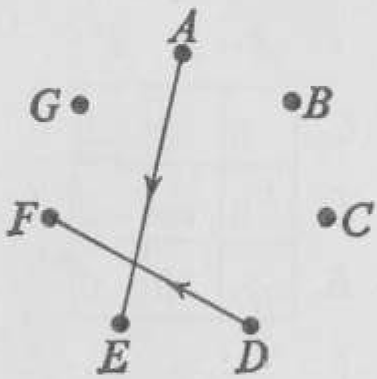
vertex ⑤′

Fig. 79

	A	B	C	D	E	G
A	∞	2	6	3	$\boxed{2}$ 0	5
B	1	∞	3	3	$\boxed{1}$ 0	∞
C	3	8	∞	$\boxed{1}$ 0	5	1
E	$\boxed{2}$ 0	4	1	$\boxed{0}$ 0	∞	5
F	1	$\boxed{0}$ 0	$\boxed{1}$ 0	∞	2	$\boxed{1}$ 0
G	5	$\boxed{2}$ 0	2	15	4	∞

vertex ⑥

Fig. 80



	A	B	C	D	G
B	1	∞	3	3	∞
C	3	8	∞	0	1
E	$\boxed{\infty}$	4	1	0	5
F	1	0	0	∞	0
G	5	0	2	15	∞

vertex ⑥'

Fig. 81

	A	*B*	*C*	*D*	*G*
B	0	∞	2	2	∞
C	3	8	∞	0	1
E	∞	4	1	0	5
F	1	0	0	∞	0
G	5	0	2	15	∞

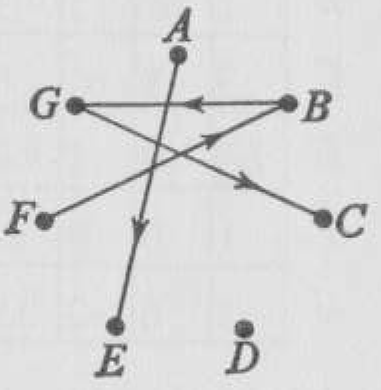
vertex 6′
Back to *FB*
(vertex 4′)

Fig. 82

	A	*C*	*D*	*F*
C	$\boxed{0}$ 0	∞	$\boxed{0}$ 0	6
D	$\boxed{0}$ 0	2	∞	$\boxed{0}$ 0
E	∞	1	$\boxed{0}$ 0	$\boxed{0}$ 0
G	$\boxed{0}$ 0	$\boxed{1}$ 0	13	∞

vertex 7

Fig. 83

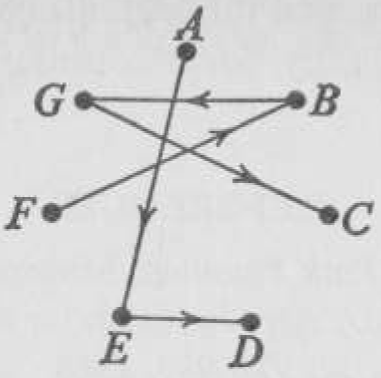


	A	*D*	*F*
C	0	0	$\boxed{\infty}$
D	0	∞	0
E	∞	0	0

vertex 7′

Fig. 84

	A	D	F
C	$\boxed{0}$ 0	$\boxed{0}$ 0	∞
D	$\boxed{0}$ 0	∞	$\boxed{0}$ 0
E	∞	$\boxed{0}$ 0	$\boxed{0}$ 0



vertex ⑧

Fig. 85

	A	F
C	0	∞
D	$\boxed{\infty}$	0

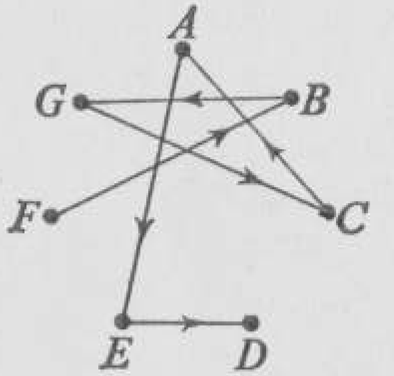
⑧'

Fig. 86

	A	F
C	$\boxed{\infty}$ 0	∞
D	∞	$\boxed{\infty}$ 0

vertex ⑨

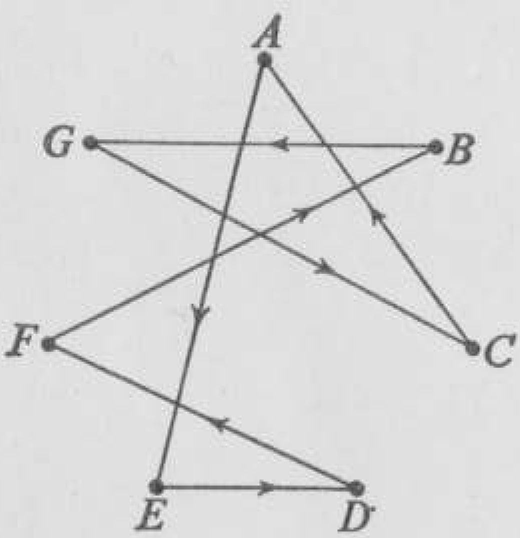
Fig. 87



	F
D	0

vertex ⑨'

Fig. 88



vertex ⑩'

Fig. 89

	A	B	C	D	E	F	G
A					3		
B							2
C	6						
D						2	
E				2			
F		2					
G			3				

Fig. 90

equally well be used to find the Hamiltonian circuit (circuits) of maximal value and finally, for a considerable number of optimisation problems.

REFERENCES

1. SMITH, K., *Critical Path Planning*, Macdonald, London, 1971.
2. LITTLE, J. D. C., et al., 'An Algorithm for the Travelling Salesman Problem', *O.R.S.A., 11*, 972–989, 1963.
3. FLEGG, H. G., *Boolean Algebra,* Transworld Student Library 1972.

6. Important Concepts Associated with a Lattice

Finite Lattice

We are going to be concerned with certain graphs endowed with a lattice structure. This structure plays a very important role both in abstract thought and in nature. Let us first recall several definitions.

Ordered Set

This is a set on which an order relation is defined. We denote it by $a \preccurlyeq b$. We will be considering here relations of non-strict order. The examples of ordered sets are innumerable: the set of integers, the set of complex numbers, a vector space, hierarchy, networks, etc.

Comparable Elements

Two elements a and b, belonging to an ordered set **E**, are said to be *comparable* if either one or the other of the following relations is true:

$$a \preccurlyeq b \qquad \text{or} \qquad b \preccurlyeq a.$$

Every element a is comparable with itself.

Minimal (Maximal) element

Let us represent an ordered finite set by a graph where, if $a \preccurlyeq b$, the graph includes the arc (a, b). Figures 91 and 92 represent ordered sets; that of figure 91 arises from a partial

order relation, whereas that of figure 92 arises from a total order relation.

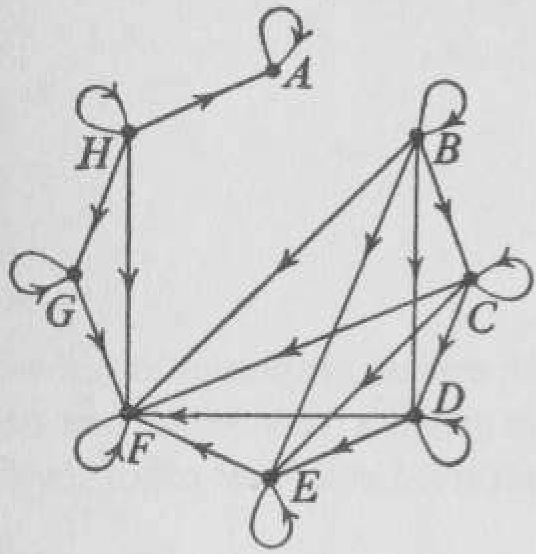


Fig. 91

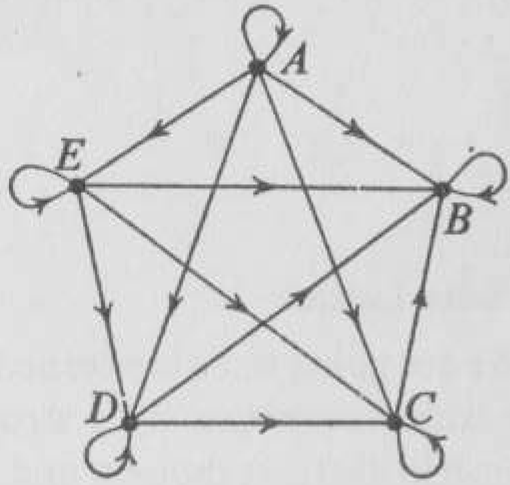


Fig. 92

An element Y of an ordered set **E** is *minimal* (*maximal*) if there are no elements of **E** which are strictly inferior (strictly superior). For example (see figure 91): B is a minimal element of the set $\mathbf{E} = \{A, B, C, D, E, F, G, H\}$. On the other hand, F is a maximal element of this set; the same applies to A.

Least Element or First Element (Greatest Element or Last Element).

An element Y of an ordered set **E** is the least (greatest) element of the set if it is less than or equal (greater than or equal) to all the other elements.

Totally Ordered Set

A set all of whose elements are comparable.

Partially Ordered Set

A set which is not totally ordered. (Not all of its elements are comparable.)

For example: the partially ordered set of figure 91 does not have a least element (B is not comparable with H); it does not have a greatest element (F is not comparable with A). On the other hand in this figure the ordered subset $\{B, C, D, E F, G, H\}$ has a greatest element, namely F, but it does not have a least element. The ordered set of figure 92 has a least element, namely A; it also has a greatest element, namely B.

Lower Bound. Upper Bound

Given $\mathbf{A} \subset \mathbf{E}$, where $\mathbf{E}$ is an ordered set, $\mathbf{A}$ is ordered by the same order relation in its elements. An element $Y \in \mathbf{E}$ is a *lower bound* (*upper bound*) of $\mathbf{A}$ if it is less than (greater than) or equal to every element of $\mathbf{A}$.

For example (see figure 91); B is a lower bound of $\{D, E, F\}$. E is an upper bound of $\{B, C, D\}$. A is an upper bound of $\{H\}$. H is a lower bound of $\{A, G, F\}$.

If $Y \in \mathbf{A}$, then the lower (upper) bound of $\mathbf{A}$ is the least (greatest) element of $\mathbf{A}$.

Greatest Lower Bound. Least Upper Bound.

Let $\mathbf{M}$ be the set of lower bounds ($\mathbf{M}'$ the set of upper bounds) of $\mathbf{A} \subset \mathbf{E}$. If $\mathbf{M}(\mathbf{M}')$ has a greatest (least) element Y, this element Y is called the *greatest lower bound* (*least upper bound*) of $\mathbf{A}$. In particular, if $\mathbf{A}$ has a least (greatest) element Y, this element Y is the greatest lower bound (least upper bound) of $\mathbf{A}$. We denote the least upper bound by *Sup* and the greatest lower bound by *Inf*.

For example (see figure 91): let us consider, in this figure, the subset $\mathbf{A} = \{D, E, F\}$; the subset $\mathbf{M} = \{B, C, D\}$ is the set of lower bounds of $\mathbf{A}$. Now, $\mathbf{M}$ has a greatest element, namely D, so we say that D is the greatest lower bound of $\mathbf{A}$. In the same way, if we consider the subset $\mathbf{B} = \{B, C, D\}$, then D is the least upper bound of $\mathbf{B}$.

Lattice

An ordered set is a lattice if every pair of elements has a greatest lower bound and a least upper bound.

For example, the graph of figure 93 is a lattice. Let us verify this: Sup $\{A, B\} = B$, Sup $\{A, C\} = B$, Sup $\{A, D\} = B$, Sup $\{A, E\} = A$, Sup $\{B, C\} = B$, Sup $\{B, D\} = B$, Sup $\{B, E\} = B$, Sup $\{C, D\} = C$, Sup $\{C, E\} = C$, Sup $\{D, E\} = D$, Inf $\{A, B\} = A$, Inf $\{A, C\} = E, \ldots,$ Inf $\{D, E\} = E$. The graph of figure 94 is not a lattice; it is, however, what we call an *inf. semi-lattice*: each pair of elements has a greatest lower bound. Similarly, the graph of figure 95 is not a lattice; it is what we call a *sup. semi-lattice*: each pair of elements has a least upper bound. In these figures, the circled vertices represent (when it exists) the least (greatest) element of the lattice or semi-lattice.

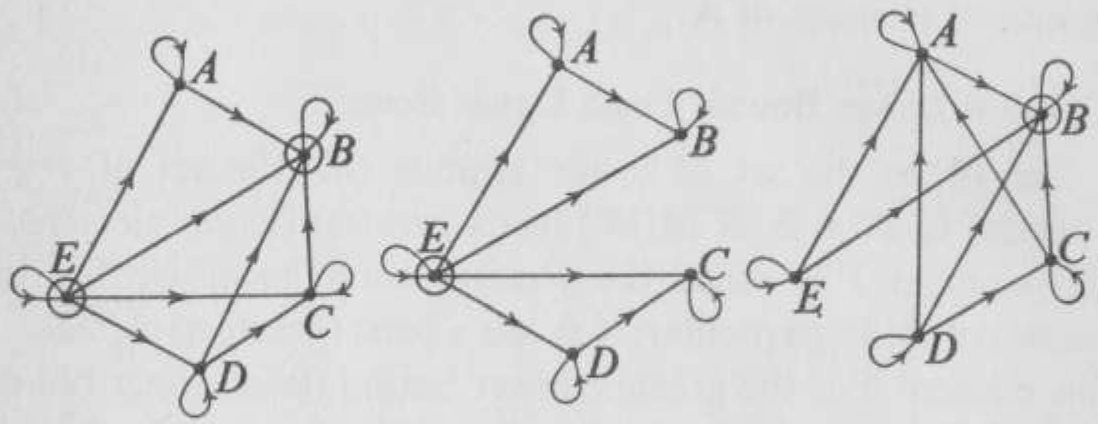


Fig. 93 Fig. 94 Fig. 95

Lattice Hasse Diagram

By convention, we will not designate the arc (X_1, X_3) when (X_1, X_2) and (X_2, X_3) belong to a graph, and we will perform such a suppression step by step, also removing the loops. We get what we call a *Hasse diagram*. Thus, the lattice of figure 93 will be represented in the way shown in figure 96. We next re-

move the arrows and put the least element at the bottom, the greatest at the top, and the others in their respective positions according to the ordering. Thus, the Hasse diagram of the lattice of figure 93 will take the form as shown in figure 97.

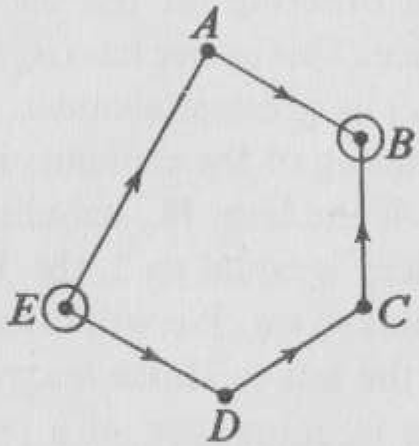


Fig. 96

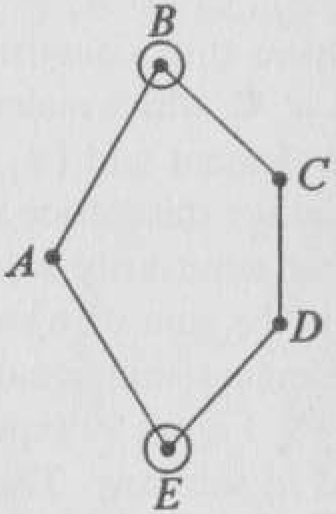


Fig. 97

Let us now show how we can introduce the concept of lattice.

Concept of Lattice and the Ordered Examination of a Field of Possibilities

Let us consider a machine, as previously, to be made up of three distinct parts α, β and γ. α can take on two variations, A_1 and A_2, forming a set $\mathbf{A} = \{A_1, A_2\}$. There are three variations for β, B_1, B_2, B_3, forming a set $\mathbf{B} = \{B_1, B_2, B_3\}$. There are two variations for γ, C_1 and C_2, forming a set $\mathbf{C} = \{C_1, C_2\}$. It is clear that we can thus form $2 \times 3 \times 2 = 12$ distinct machines.

(A_1, B_1, C_1), (A_1, B_1, C_2), (A_1, B_2, C_1), (A_2, B_1, C_1), (A_1, B_2, C_2), (A_1, B_3, C_1), (A_2, B_1, C_2), (A_2, B_2, C_1), (A_1, B_3, C_2), (A_2, B_2, C_2), (A_2, B_3, C_1) and (A_2, B_3, C_2).

Let us suppose then that each of the sets, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, are totally and strictly ordered by the suffices of their elements;

that is to say: $A_1 \succ A_2$, $B_1 \succ B_2 \succ B_3$, $C_1 \succ C_2$. If we consider the set product $\mathbf{A} \times \mathbf{B} \times \mathbf{C}$, that is to say, the set of 12 triples (A_i, B_j, C_k); $i = 1, 2$; $j = 1, 2, 3$; $k = 1, 2$, we can construct an order on these triples by deciding that $(A_i, B_j, C_k) \succcurlyeq (A_{i'}, B_{j'}, C_{k'})$ if $A_i \succcurlyeq A_{i'}$, $B_j \succcurlyeq B_{j'}$, $C_k \succcurlyeq C_{k'}$.

We have thus constructed an ordering on the set product $\mathbf{A} \times \mathbf{B} \times \mathbf{C}$, which makes a lattice. This lattice has (A_2, B_3, C_2) as least element and (A_1, B_1, C_1) as greatest element. We can construct for this lattice a positioning of the elements in levels, numbered arbitrarily from 0 to 4, the level $\mathbf{N}_0$ containing the elements the sum of whose suffices is equal to 3, the level $\mathbf{N}_1$ for an element-sum equal to 4, and so on. We will form figures 98 and 99. Figure 99 represents the lattice Hasse diagram constructed in this way. This lattice is, moreover, of a particular type formed by a superposition of hypercubes (in this example, two cubes are involved). The example which we have given is only a particularly simple case of much more general combinatorial problems where we examine the domain of all possibilities and where we introduce a structure of order on the elements. We imagine encountering this very often in nature and in the abstract. In particular, this concept is used in the exploration of possibilities in morphological study and in industrial value-analysis, where we wish to select machines or complex structures according to different criteria.

In figures 100 to 103, we have given the Hasse diagrams corresponding to lattices called *Boolean lattices*. They relate to set products where each set involved has exactly two elements, for examle 0 and 1. These Boolean lattice structures form hypercubes with the help of their Hasse diagrams. (The relationship between Boolean algebra and lattice theory is explained in reference (1) at the end of this chapter.)

In figure 104, we have presented another example belonging to the same family of lattices as that of figure 99. This time, we

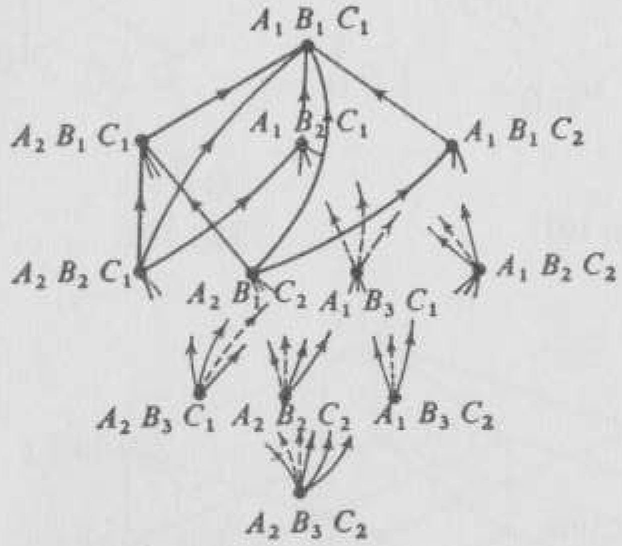

$A_1 B_1 C_1$
$A_2 B_1 C_1$
$A_1 B_2 C_1$
$A_1 B_1 C_2$
$A_2 B_2 C_1$
$A_2 B_1 C_2$
$A_1 B_3 C_1$
$A_1 B_2 C_2$
$A_2 B_3 C_1$
$A_2 B_2 C_2$
$A_1 B_3 C_2$
$A_2 B_3 C_2$

Fig. 98

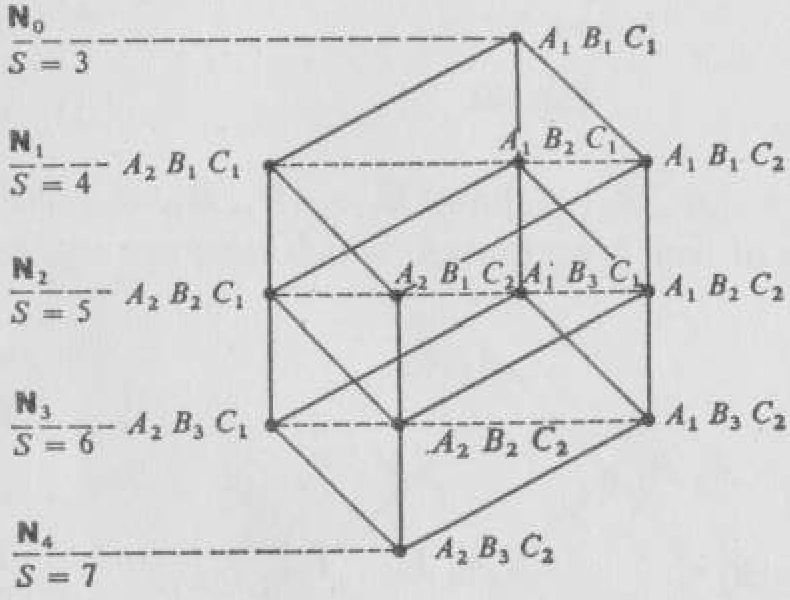

$\frac{\mathbf{N}_0}{S = 3}$
$A_1 B_1 C_1$
$\frac{\mathbf{N}_1}{S = 4}$
$A_2 B_1 C_1$
$A_1 B_2 C_1$
$A_1 B_1 C_2$
$\frac{\mathbf{N}_2}{S = 5}$
$A_2 B_2 C_1$
$A_2 B_1 C_2$
$A_1 B_3 C_1$
$A_1 B_2 C_2$
$\frac{\mathbf{N}_3}{S = 6}$
$A_2 B_3 C_1$
$A_2 B_2 C_2$
$A_1 B_3 C_2$
$\frac{\mathbf{N}_4}{S = 7}$
$A_2 B_3 C_2$

Fig. 99

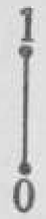


Fig. 100

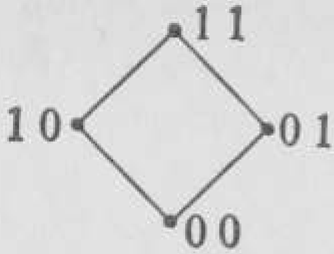


Fig. 101

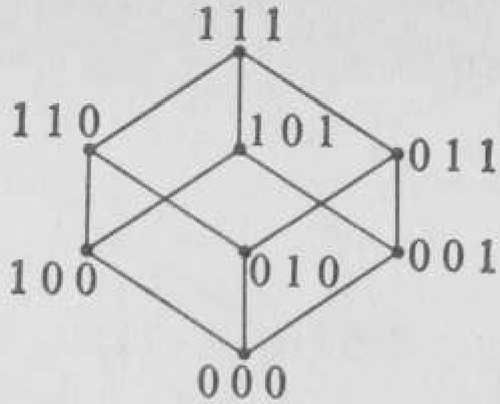


Fig. 102

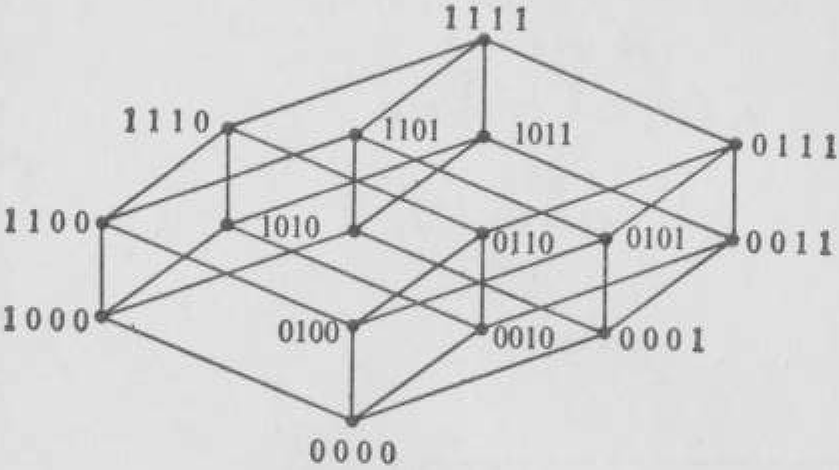


Fig. 103

take two sets $\mathbf{A} = \{A_1, A_2, A_3\}$ and $\mathbf{B} = \{B_1, B_2, B_3\}$ and we find an assembly of four hypercubes, which here are squares.

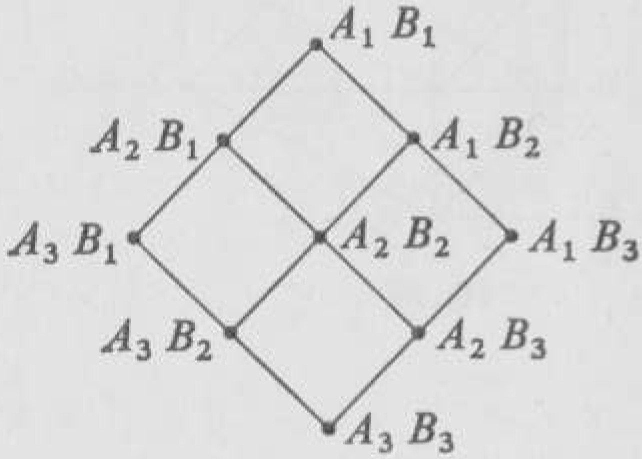


Fig. 104

Directed Tree

We call a *directed tree* a circuitless graph such that:

(1) there is one and only one vertex where no arc finishes, this vertex being called the *root*;

(2) one and only one arc finishes at each other vertex.

Figure 105 represents a directed graph whose root is A.

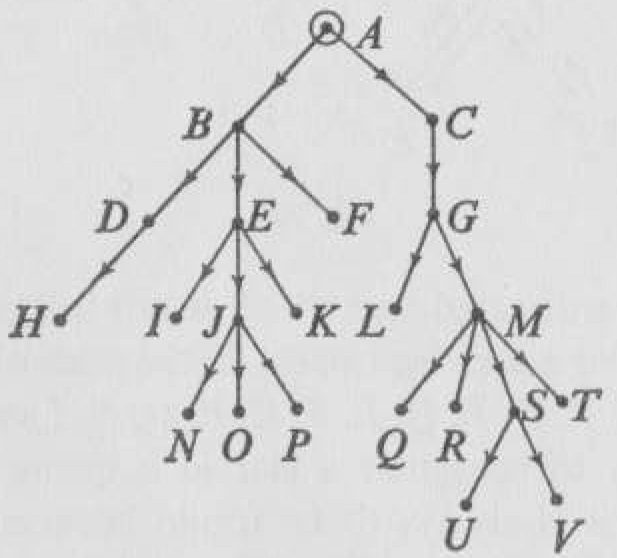


Fig. 105

The most classical example concerning this concept is that of the genealogical tree. If we consider male descendants in a family, each arc represents the relation 'is the son of'. If we consider the relation 'is descended from', the directed tree becomes an inf. semi-lattice.

Amongst other interesting questions is that of seeking how to describe a directed tree by a sequence of vertices of a graph, the concept of *queue* explained with an easy suitable example. Consider the example of figure 106. Let us surround the directed tree by a dotted envelope (as shown in the figure) thus obtaining the arrows:

$$(C, B, A, I, A, H, A, D, A, B, F, G, E, G, J, G, F, B, C);$$

the sequence perfectly defines the directed tree.

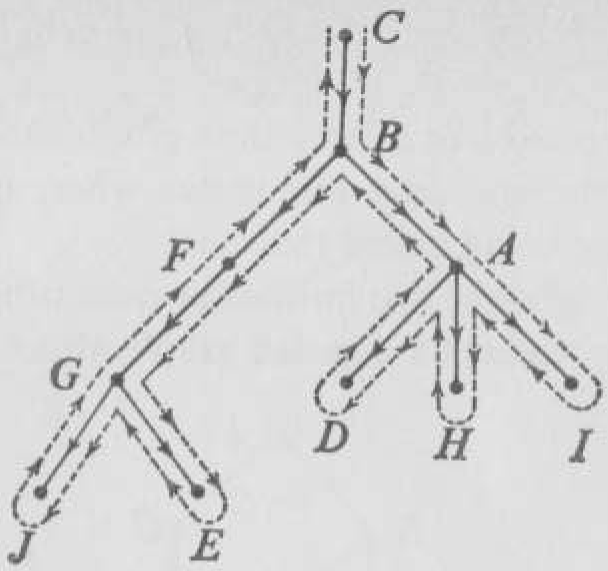


Fig. 106

A vertex of a directed tree which is not followed by another is called a *leaf* or a *hanging vertex*. In the example of figure 105, the vertices, T, V, U, R, Q, L, F, K, P, O, N, I and H are leaves. It is very easy to recognise a leaf in a queue representing a directed tree; it is always to be found between two repeated vertices. Thus, for the example of figure 106, I is between A and A, H is between A and A, also D; E is between G and G, also J.

This idea of a queue due to Pair[2] is particularly important for the computerising of a directed tree†; it is easier to store a queue and to handle it sequentially than it is to store and handle a matrix.

We call a set of directed trees a *branching*. For example, see the graph of figure 107.

If we consider any graph whatsoever, it is possible in certain

† In another code given by Pair, we can describe the directed tree of figure 106 as:

$$
\begin{array}{c}
E \quad J \\
I \quad H \quad D \qquad G \; G \; G \; G \; G \\
A \; A \; A \; A \; A \; A \; A \qquad F \; F \; F \; F \; F \; F \; F \\
B \; B \; B \; B \; B \; B \; B \; B \; B \; B \; B \; B \; B \; B \; B \; B \; B \\
C \; C \; C \; C \; C \; C \; C \; C \; C \; C \; C \; C \; C \; C \; C \; C \; C \; C \; C \; C \; C
\end{array}
$$

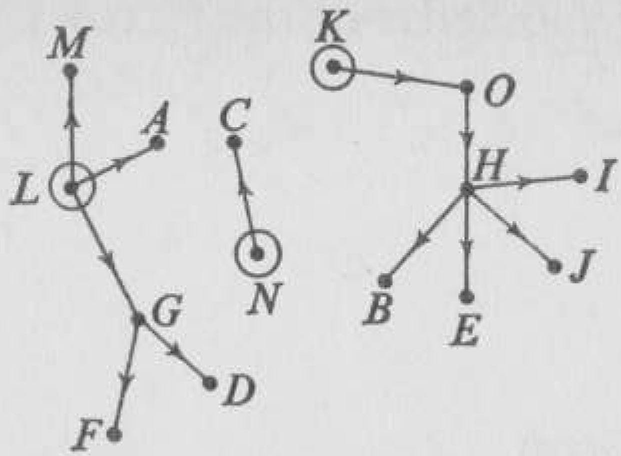


Fig. 107

cases to find in the graph a partial graph which is a directed tree. We say then that we have obtained a *partial directed tree* of the graph. For example, see figures 108 and 109. The distinguishing and enumeration of partial directed trees are important problems which we can find in both pure and applied mathematical problems (coding, linguistics, translation, etc.).

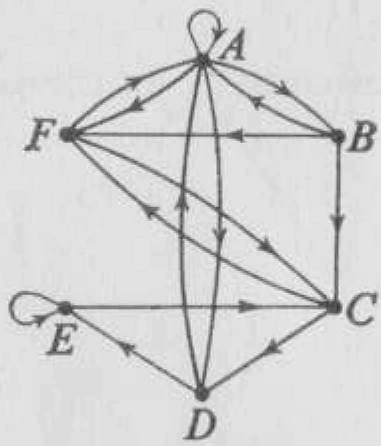


Fig. 108

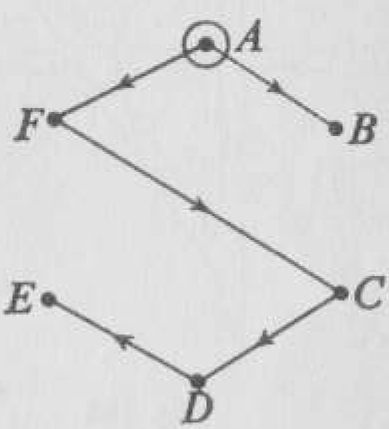


Fig. 109

REFERENCES

1. Flegg, H. G., *Boolean Algebra*, Transworld, London, 1972.
2. Pair, C., *Etude de la notion de pile. Application à l'analyse synthaxique*. Thèse Fac. Sciences, Nancy, 1966.

7. Ignoring the Sense of the Arrows

Non-oriented Graph

Consider a set of junctions and streets in a town (see figure 110). For a motorist, the graph of figure 111 will provide a good representation taking account of two-way and one-way streets. But for a pedestrian, who does not take account of the particular

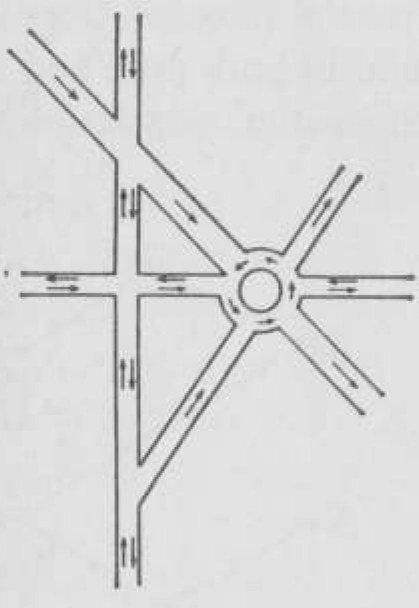


Fig. 110

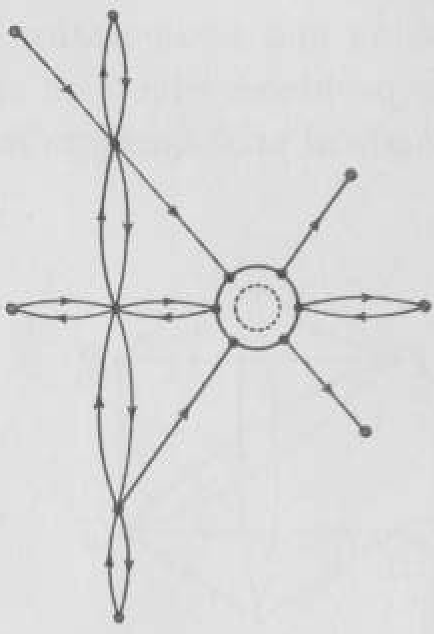


Fig. 111

directions (not so far in our age), a representation giving the streets without directional constraint (see figure 112) is sufficient. In this way, a concept, which is not a graph but which we nevertheless call a *non-oriented graph*, can be associated with every graph.

Another example of a non-oriented graph (see figure 114)

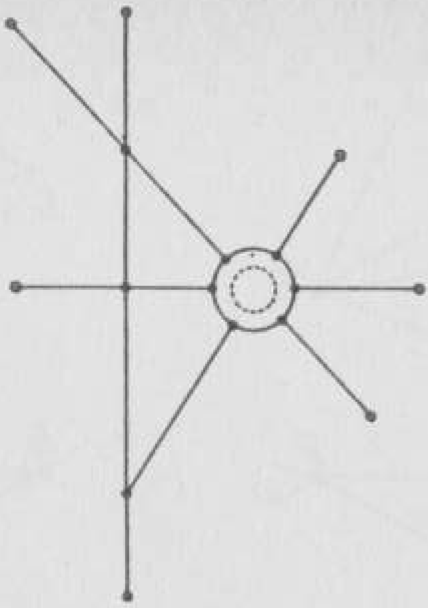


Fig. 112

associated with a graph (see figure 113) is given below. We ought not to confuse a symmetric graph, such as that of figure 115, with the non-oriented graph (see figure 114) which might seem to be equivalent. Returning to the example of figure 110, the graph to be considered by a motorist is that of figure 111; for a road-maker constructing the streets the graph to be considered would be that of figure 112; for a pedestrian it could equally well be either the non-oriented graph or the symmetric graph which would be taken into account.

On the other hand, the concept of a non-oriented graph can be defined independently from that of a graph (as certain American authors have done). It all depends on the way in which the definitions are given.

Branch of a Graph

A pair of distinct elements X_i and X_j, to which at least one of the arcs (X_i, X_j) or (X_j, X_i) corresponds, is called a *branch* of a graph $G = (\mathbf{E}, \mathbf{U})$, where $\mathbf{U}$ is the set of arcs. In other

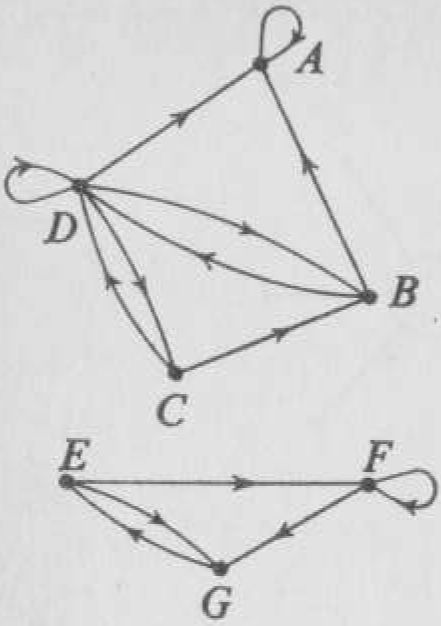


Fig. 113

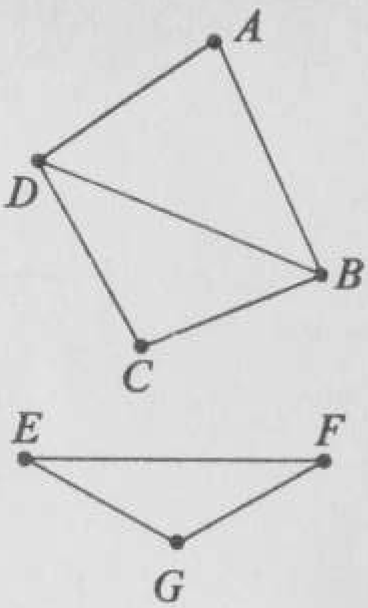


Fig. 114

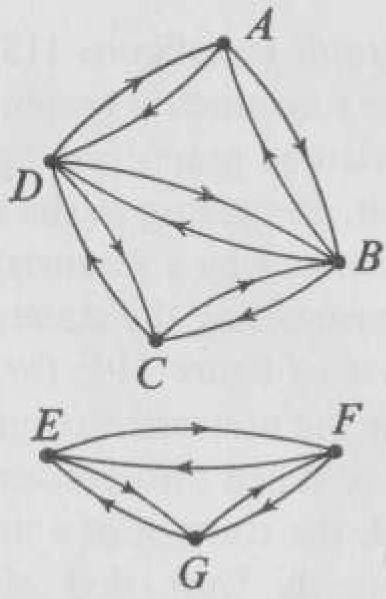


Fig. 115

words, a branch is a pair of vertices linked by an arc going one way or the other, or by two arcs going in opposite directions.

For example, in figure 113 there is a branch between A and B, a branch between E and G, but there is no branch between A and C.

A branch is denoted by

$$\bar{u} = (\overline{X_i, X_j}),$$

or by

$$\bar{u} = [X_i, X_j].$$

The set of branches of a graph is denoted by $\overline{\mathbf{U}}$.

The concept defined by the ordered pair $(\mathbf{E}, \overline{\mathbf{U}})$ is called a *non-oriented graph* and is designated by:

$$\overline{G} = (\mathbf{E}, \overline{\mathbf{U}}).$$

One and one only non-oriented graph $\overline{G} = (\mathbf{E}, \overline{\mathbf{U}})$ is associated with a graph $G = (\mathbf{E}, \mathbf{U})$, the converse obviously not being true.

Chain. Cycle.

A chain is a sequence of branches $(\bar{u}_1, \bar{u}_2, \ldots)$; each branch $\bar{u}_k$ is joined at one end to $\bar{u}_{k-1}$ (if it exists) and at the other to $\bar{u}_{k+1}$ (if it exists). A chain is denoted by the vertices which it includes or by its branches. A chain can be finite or not.

The length of a chain is the number of branches which it contains.

For example, in figure 113, $(\overline{A, B, C, D, B})$ and $(\overline{E, F, G})$ are chains.

The adjectives 'elementary' and 'simple' apply to chains in the same way that they have been applied to paths.

A cycle is a closed chain.

Connected Graph

A graph G is connected if there is a chain between each pair of vertices. The graph of figure 111 is connected, that of figure 113 is not.

A strongly connected graph is connected but, obviously, the converse is not true.

A non-connected graph divides up into connected components. The graph of figure 113 has two connected components.

Degree of a Vertex

This is the number of branches which have their extremities at the vertex. Thus, in figure 114, the degree of A is 2, and that of D is 3.

Chain and Hamiltonian cycle

These definitions are the same as for paths and circuits except that we consider branches and not arcs.

Tree. Partial Tree of a Graph

A *tree* is a non-oriented finite graph $\bar{G}$ having no cycle and having at least two vertices. Figure 116 represents a tree.

A partial graph $\bar{H}$, which is a tree, is called a *partial tree* of a graph $\bar{G}$. (We no longer write *non-oriented graph* where we use the designation $\bar{G}$.) Thus, in figure 117, we have shown a partial tree of a given graph $\bar{G}$ with heavy lines. We can in general determine several partial trees in a graph $\bar{G}$, and in

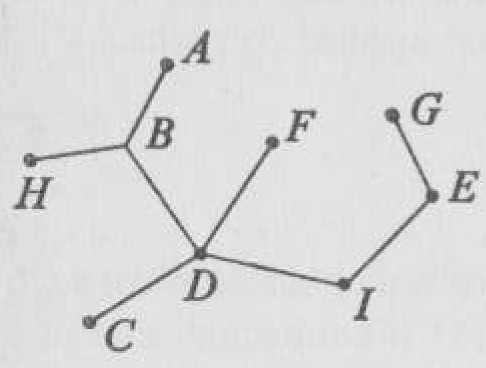


Fig. 116.

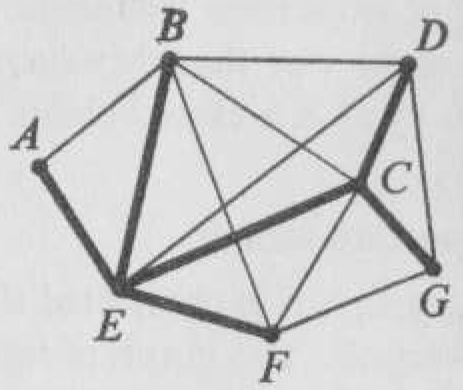


Fig. 117

various problems we will be wanting to determine the best tree according to certain criteria. (We will be studying an optimal tree problem a little later on.)

Articulation Point and Articulation Subset

The vertex P of a connected graph $\bar{G}$ is an *articulation point* if the graph $\bar{G}'$, obtained from $\bar{G}$ be removing P and the branches connected to it, is no longer connected. In figure 118, the vertices C, D and E are articulation points.

If X and Y are two vertices belonging to two connected components of $\bar{G}'$, every chain of $\bar{G}$ passing through X and Y must necessarily pass through P.

This concept can be extended to that of an *articulation subset*. Thus, in figure 119, if we remove the vertices C and D, we obtain two subsets forming a non-connected graph.

An interesting problem is that of the search for an articulation subset (or subsets) comprising a minimal number of vertices.

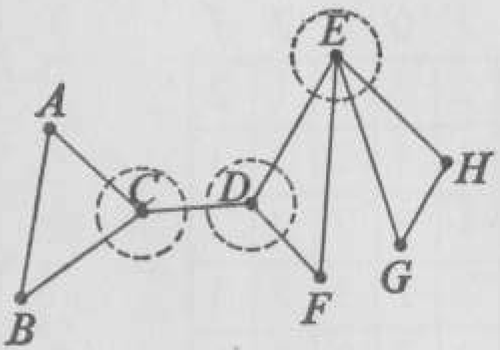


Fig. 118

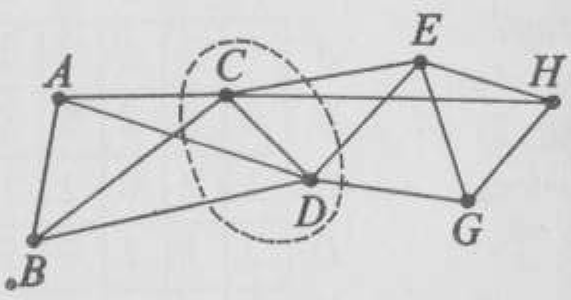


Fig. 119

Distance

The *distance* between two vertices X_i and X_j of a connected graph $\bar{G}$ is the length of the shortest chain between X_i and X_j.

We will denote this distance by $\delta(X_i, X_j)$.

For example, in figure 120, $\delta(D, G) = 2$, $\delta(D, J) = 3$.

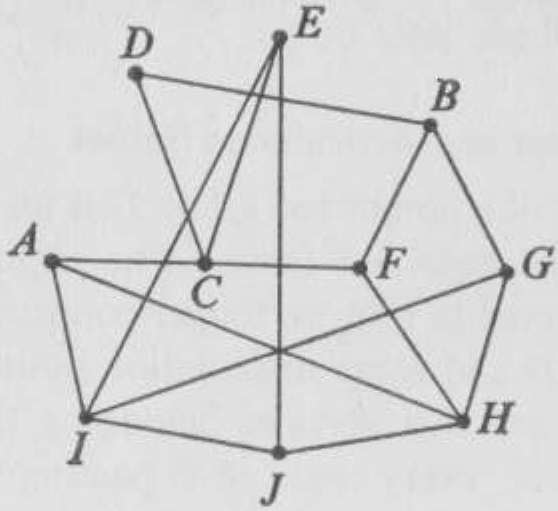


Fig. 120

Diameter

The *diameter* of a graph $\bar{G}$ is the greatest distance which can be found in the graph. For example, the table, given below, gives the distances. We see that the diameter of the graph is 3.

	A	B	C	D	E	F	G	H	I	J
A	0	3	1	2	2	2	2	1	1	2
B	3	0	2	1	3	1	1	2	2	3
C	1	2	0	1	1	1	3	2	2	2
D	2	1	1	0	2	2	2	3	3	3
E	2	3	1	2	0	2	2	2	1	1
F	2	1	1	2	2	0	2	1	3	2
G	2	1	3	2	2	2	0	1	1	2
H	1	2	2	3	2	1	1	0	2	1
I	1	2	2	3	1	3	1	2	0	1
J	2	3	2	3	1	2	2	1	1	0

Deviation of a Vertex

The greatest distance which a vertex X_i has with respect to any of the other vertices is called the *deviation* of the vertex X_i. We denote it by $e(X_i)$; we thus have

$$(27) \qquad e(X_i) = \max_{X_j \in \mathbf{E}} \delta(X_i, X_j).$$

Thus, in the example given by figure 120, $e(G) = 3$. We note that this graph is a special case where all the vertices have the same deviation 3.

Centre

If $\min_{X_i \in \mathbf{E}} e(X_i)$ is a finite number, a vertex with minimal deviation is a centre. In our example (figure 120), every vertex is a centre.

Peripheral Point

This is the vertex of maximal deviation. In our example (figure 120), every vertex is at the same time a centre and a peripheral point. This indicates a very special structure. We are going to see a little later on other examples where the vertices are arranged very differently.

Euler Chain. Euler Cycle

A chain (or a cycle) is an *Euler chain* (*cycle*) if it makes use of each branch once and once only; in other words, if we can trace the non-oriented graph G without raising our pen and without passing twice over the same line. (The reader should make up an example for himself.)

Examples of Using Non-oriented Ideas

Various psychologists and mathematicians, particularly Leavitt[1] and Harary[2,3], have been interested in the structure of

networks of communications between individuals and have sought in these networks various shapes and numerical scales especially significant to a psychologist. These are, for example, the presence of minimal articulation subsets, and the centrality or remoteness of members of a group of human beings having possibilities of cummunications. Thus, let us take four kinds of communications networks each invovling five persons (see figure 121); in this figure we have also given the tables of distances.

Leavitt has introduced a *centrality index* β_i for each vertex X_i of a non-oriented graph, thus:

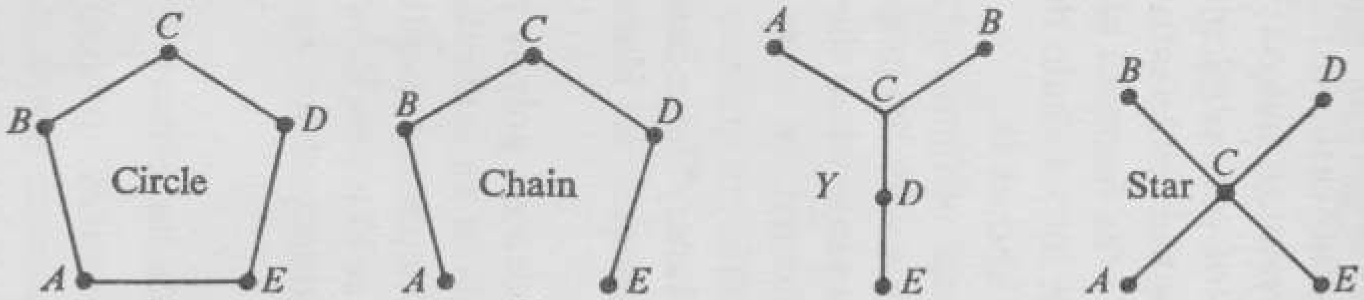
$$\beta_i = \frac{\sum_i \sum_j \delta(X_i, X_j)}{\sum_j \delta(X_i, X_j)}, \tag{28}$$

that is to say, an index obtained by summing the elements of the distance table and dividing the result by the sum of the distances of X_i from the other vertices (the sum of the elements of row i in the distance table). He has also introduced a *peripherality index* γ_i for each vertex X_i, which is equal to the difference between the value β_k of a centre X_k and the value β_i of the vertex X_i. We thus have for the four graphs of figure 121:

	Circle		Chain		Y		Star	
	β_i	γ_i	β_i	γ_i	β_i	γ_i	β_i	γ_i
A	5	0	4	2.66	4.5	2.7	4.57	3.43
B	5	0	5.71	0.95	4.5	2.7	4.57	3.43
C	5	0	6.66	0	7.2	0	8	0
D	5	0	5.71	0.95	6	1.2	4.57	3.43
E	5	0	4	2.66	4	3.2	4.57	3.43

From these results, it is possible to supply interesting information about the communications networks under con-

	Circle	Chain	Y	Star
Articulation point	any	*B*, *C* and *D*	*C* and *D*	*C*
Centre	any (or all)	*C*	*C* and *D*	*C*
Peripheral vertices	any (or all)	*A* and *E*	*A*, *B* and *E*	*A*, *B*, *D* and *E*



(a) (b) (c) (d)

	A	B	C	D	E
A	0	1	2	2	1
B	1	0	1	2	2
C	2	1	0	1	2
D	2	2	1	0	1
E	1	2	2	1	0

	A	B	C	D	E
A	0	1	2	3	4
B	1	0	1	2	3
C	2	1	0	1	2
D	3	2	1	0	1
E	4	3	2	1	0

	A	B	C	D	E
A	0	2	1	2	3
B	2	0	1	2	3
C	1	1	0	1	2
D	2	2	1	0	1
E	3	3	2	1	0

	A	B	C	D	E
A	0	2	1	2	2
B	2	0	1	2	2
C	1	1	0	1	1
D	2	2	1	0	2
E	2	2	1	2	0

Fig. 121

sideration. The star arrangement is clearly the most centralised, that in a circle is equally clearly the least centralised; the removal of C from the star would prevent all liaison; in the circle, the removal of a possible means of communication, for example that from A to E, transforms the circle into a chain with an increase in the centrality of C; the removal of an element such as A also transforms the circle into a chain but with a relative increase of the centrality of C and of D.

For graphs as simple as those of figure 121 the conclusions which we have been able to draw with the help of the theory are virtually obvious; it is not the same when the communications network is much more complicated. In order to determine the articulation points and the different indices we make use of matrix calculation. Ross and Harary[4] have dealt with numerous properties. Our previous references to Harary[2, 3] may also be consulted with interest.

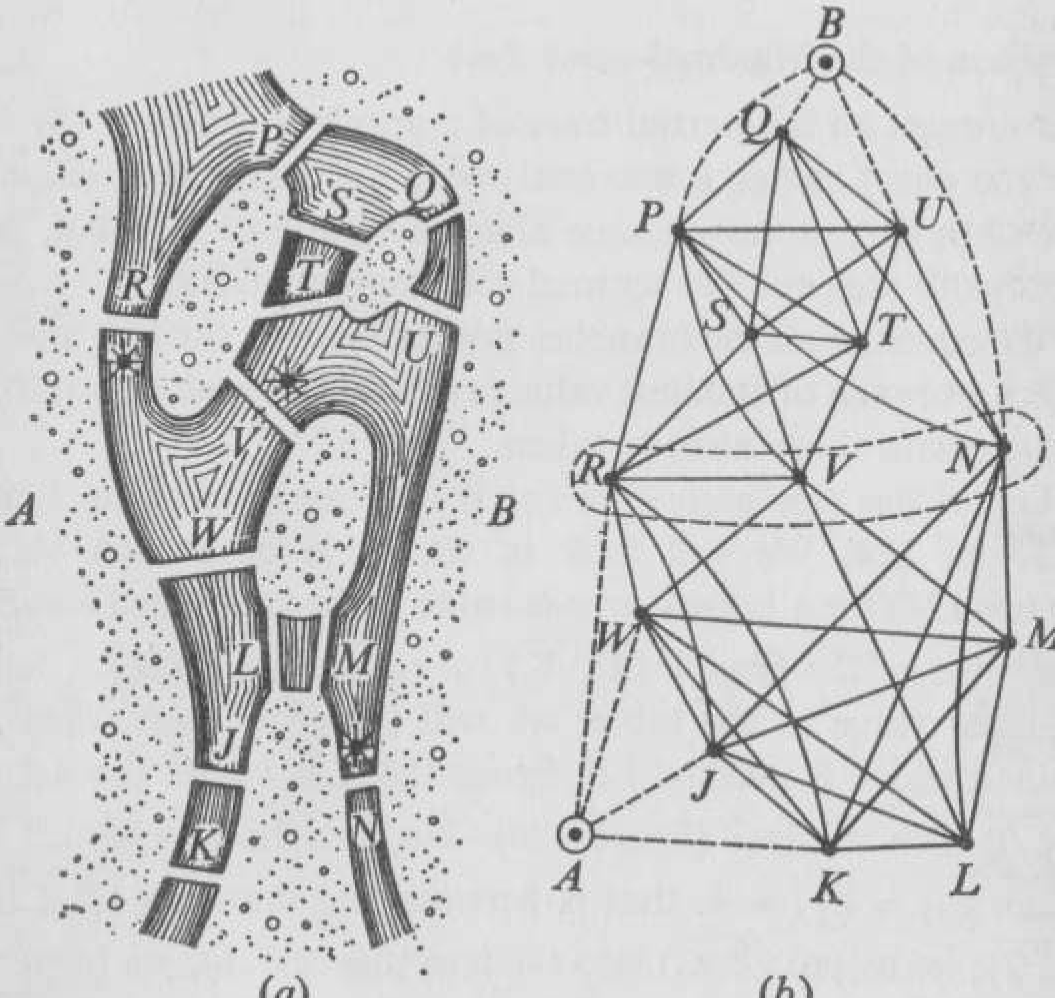
From the theory of non-oriented graphs psychologists can determine in what way the role of an individual is crucial (this occurs when we find an articulation point having a large centrality), what is the influence of a new liaison or the removal of certain others, what redundancy effects there are on cycles, and so on.

Looking for a Minimal Articulation Subset

This is a subset (not necessarily unique) of the minimal number of vertices in a connected graph $\overline{G}$ which must be removed in order to separate it into two graphs $\overline{G}_1$ and $\overline{G}_2$, not connected the one with the other. An example given by Berge[5] gives an idea of the interest in the search for a minimal articulation subset.

We consider the communications (roads, railways, etc.) which exist between different regions and we want to destroy the centres of communication between the regions concerned.

For example, in the map of figure 122, we look for the minimal number of bridges which must be destroyed in order to make communications between the regions A and B on either side of the river impossible. We transform the figure thus obtaining a non-oriented graph which represents the existing links between the different points. Thus, since we can go from bridge R to bridge S, we will consider that there is a branch between R and S, the same between S and U, and so on; this is not the case for R and M or for U and V, etc. We also consider A and B as vertices which can be joined by the branches at the vertices representing the bridges having one of their entrances in these respective regions.



(a) (b)

Fig. 122

The problem is then reduced to finding in the non-oriented graph of figure 122 a subset of the minimal number of vertices which, when removed, divide the graph into two disjoint subgraphs, one containing A, the other containing B. There are various procedures for carrying out such a study (see, for example, the work of Berge[5] and that of Kaufmann[6]). We find that in destroying N, R and V we separate the regions A and B from each other, and there are then no further possibilities of communication. The articulation subset $\{N, R, V\}$ is minimal; it is not possible to separate the two regions by destroying less than three bridges. Such problems can also arise when we look for the sensitive components of an organisation, delivery network, telecommunications system, and so on.

Problem of the Maximal-value Tree

Amongst all the partial trees of a graph $\bar{G}$, which is the one (or the ones) having a maximal value if we allocate a value to each branch? A very simple algorithm, due to Kruskal, permits us to seek for the optimal solution (solutions).

We consider all the branches not included in the tree and we select a branch of smallest value provided that it does not form a cycle with those already taken.

Let us use this algorithm for the example given in figures 123 and 124. We will look in the table of branch values (figure 124) for a branch whose value is the smallest. Denoting the value of the branch $\overline{(X_i, X_j)}$ by c_{ij}, we find that c_{14} has a minimal value in the table; we will therefore start with this branch which is marked I in figure 124. Let us put the branch $\overline{(A, D)}$ into the tree (figure 126). We then have a choice between $c_{23} = c_{27} = 4$, that is between the branches $\overline{(B, C)}$ or $\overline{(B, G)}$; let us put $\overline{(B, C)}$ into the tree (figure 126); we have not formed a cycle so we put II in the 2, 3 cell. Let us put $\overline{(B, G)}$

into the tree (figure 126); we have not formed a cycle so we put III in the 2, 7 cell. We then record 6, 7, but we cannot record 3, 7 as this would form a cycle. We continue with 2, 4, which will be recorded; we discard 3, 5 (cycle) and we record 5, 7. We cannot record 1, 2 or 1, 3 (cycle). The optimal tree is given finally in figure 126; its value is:

$$3 + 4 + 4 + 5 + 7 + 8 = 31.$$

The optimal solution is unique in this example, which is a special case.

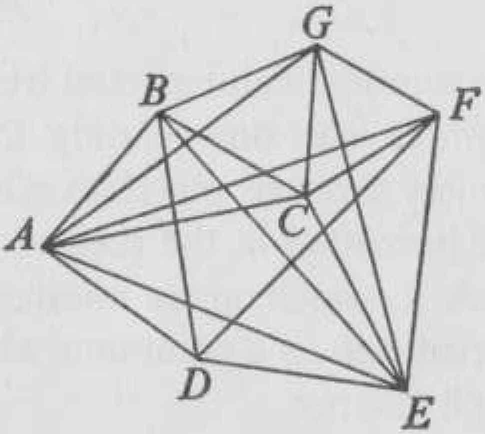
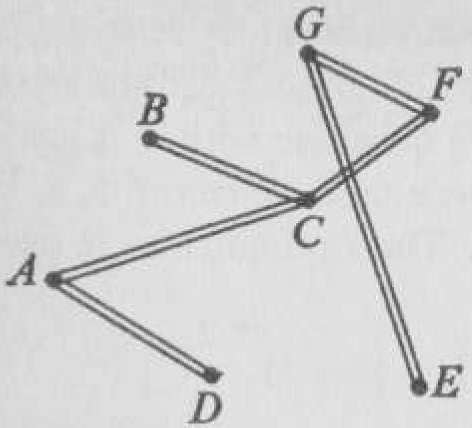


Fig. 123

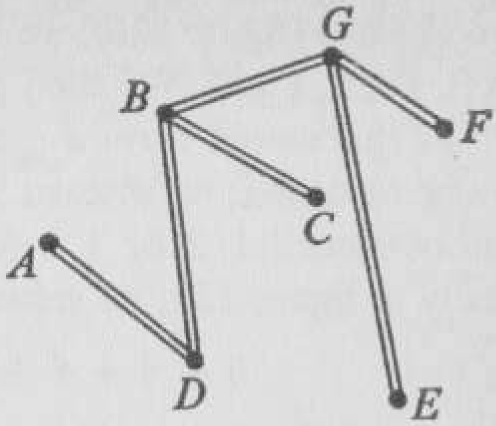
		1	2	3	4	5	6	7
		A	*B*	*C*	*D*	*E*	*F*	*G*
1	*A*	0	8	8	3_{I}	11	10	9
2	*B*	8	0	4_{II}	7_{V}	12	∞	4_{III}
3	*C*	8	4	0	∞	9	7	6
4	*D*	3	7	∞	0	13	9	∞
5	*E*	11	12	9	13	0	9	8_{VI}
6	*F*	10	∞	7	9	9	0	5_{IV}
7	*G*	9	4	6	∞	8	5	0

Fig. 124



Any partial tree

Fig. 125



Optimal partial tree

Fig. 126

The search for a minimal value partial tree arises in various problems: for example, the one relating to the supply of a network of consumers through pipelines. On this subject, see the work of Sollin (recorded in the present author's book on operational research[7]), which gives another method of calculation easily carried out on a computer when the number of vertices of the graph is large.

Problem of the Colouring of a Geographical Map. Chromatic Number

Let us consider a map of the departments of France or the states in the USA. What is the minimal number of different colours to be used so that we never have two neighbouring regions having the same colour? This is the problem of the colouring of a geographical map.

We say that a non-oriented graph is *r-chromatic* if it is possible to colour its vertices with r distinct colours such that two adjacent vertices never have the same colour. The smallest number for which the graph is r-chromatic is called the *chromatic number* of the graph $\overline{G}$.

For example, let us consider the map of ten regions represented in figure 127. It can be coloured with four arbitrary

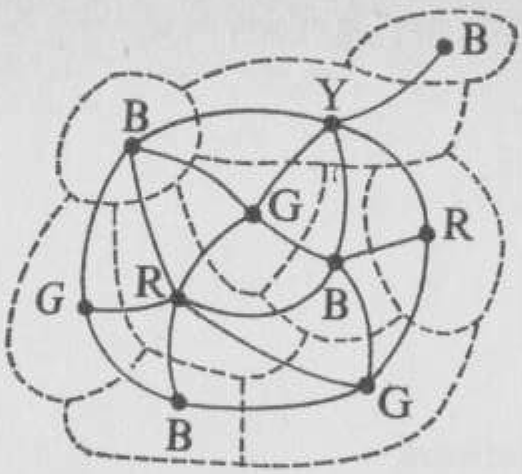


Fig. 127

colours: blue (B), yellow (Y), green (G) and red (R). We can verify that it is not possible to do it with less than four colours if we want no region to have the same colour as one of its neighbours. There are various algorithms for determining the chromatic number of a graph (see reference (6) below).

REFERENCES

1. Leavitt, H. J., 'Some Effect of Certain Communications Patterns on Group Performance', *J. Abnormal and Social Psychology*, 1951, 46–50.
2. Harary, F., and Norman, R. Z., *Graph Theory as a Mathematical Model in Social Science*. University of Michigan, Institute for Social Research, Ann Arbor, 1963.
3. Harary, F., Norman, R. Z., and Cartwright, D., *Structural Models. An Introduction to the Theory of Directed Graphs*. Wiley, 1965.
4. Ross, I. C. and Harary, F., 'On the determination of Redundancies in Sociometric Chains', *Psychometrika*, 1952, 195–208.
5. Berge, C., *La Théorie des Graphes et ses Applications*. 2nd Edition. Dunod, Paris, 1966.
6. Kaufmann, A., *Introduction à la Combinatorique en vue des Applications*. Dunod, Paris, 1968.
7. Kaufmann, A., *Méthodes et Modèles de la Recherche Opérationelle* Volume 2. Dunod, Paris, 1968.

8. How Saturation Arises

Flow through a Network

Consider a connected graph G without loops, which has one and only one vertex at which no arc terminates and which we shall call the *inlet*, and one and only one vertex at which no arc commences and which we shall call the *outlet*. Suppose that some substance circulates in the arcs of this graph and that the limiting capacity of each arc is given and cannot be exceeded by the amount of substance flowing through the arc. Such a graph is called a *transport network*, An example is given in figure 128.

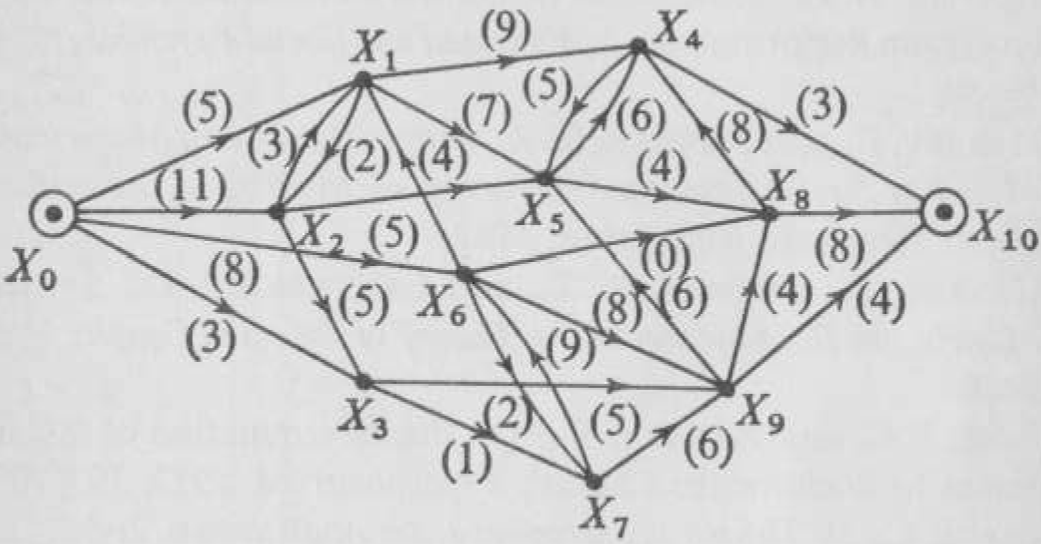


Fig. 128

We want, therefore, to determine what quantity of substance we can have passing through each arc in such a way that the

total flow from inlet to outlet in the network is maximal. We require further that for every vertex other than the inlet and outlet the amount of substance entering is equal to that leaving (conservation of flow at each vertex).

A very simple example of a transport network is given by the maritime traffic between two continents. Given m ports of departure where there are x_i ($i = 1, 2, \ldots, m$), facilities and n ports of arrival where the requirements are y_j ($j = 1, 2, \ldots, n$). The limiting transport capacities between the departure points and the arrival points are given and denoted by c_{ij}, where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. (These c_{ij} are clearly non-negative numbers, and, if one of the numbers is zero, this signifies that the corresponding traffic is not possible.) We require, then, to determine flows, $r_{ij} \leqslant c_{ij}$, for each connection such that the overall transportation is maximal. We have, evidently, the constraints

$$\sum_{j=1}^{n} r_{ij} \leqslant x_i \quad \text{and} \quad \sum_{i=1}^{m} r_{ij} \leqslant y_j.$$

In collecting all the departure points in a single point E and all the arrival points in a single point S (see figure 129), we obtain a transport network.

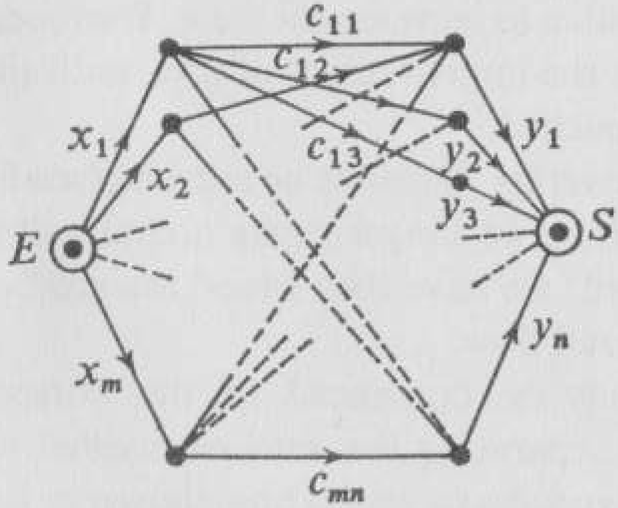


Fig. 129

Another kind of transportation problem consists in associating with each arc, between each departure point and each arrival point, a transport cost γ_{ij} (infinite in value if the corresponding transportation is impossible); we then look for an overall minimal-cost flow.

We shall concern ourselves here with the first of these transportation problems.

We know various algorithms for finding a maximal flow in a transport network, the most used of which is that of Ford-Fulkerson[1]. We are now, summarily, going to show its use with an example (see figure 130).

1. Guess a flow; this is what we have done in figure 131.
2. Put a + at the inlet E, then mark the vertices of the network successively according to the following criteria:
   (a) If a vertex is at the end of a non-saturated arc (that is to say, has a flow strictly less than its capacity) whose beginning is marked, then we mark this vertex.
   (b) We mark in the same way the beginnings of arcs, having non-zero flow, whose ends are marked.
3. If we thus succeed in marking the outlet S with a +, then we can obtain a better overall flow (see figure 132) by considering first the paths then the chains from E to S along which it is possible to increase the flow. We repeat the marking operations and the improvement of flow until the vertex S can no longer be reached.
4. When the vertex S cannot be marked (see figure 133), this signifies that there no longer exists a chain along which flow can be improved; we have then found the arc-flows which give a maximal overall flow.

We can easily be convinced of the correctness of this affirmation by separating the vertices marked with a + from the unmarked vertices (a separation shown in figure 133). The arcs which end in the set of unmarked vertices are all saturated,

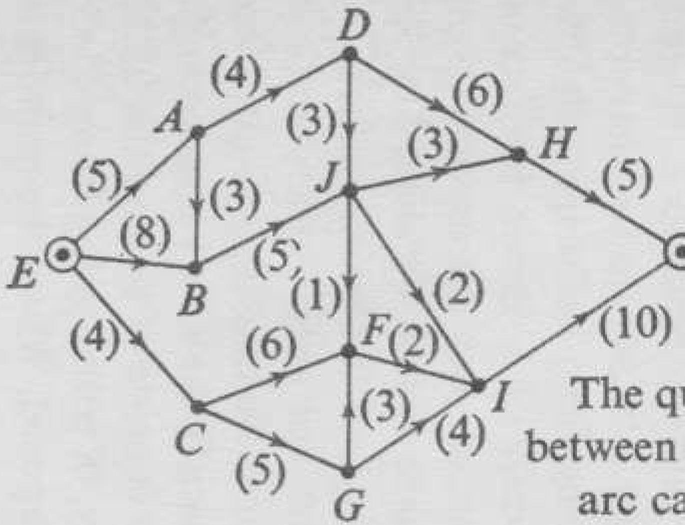


Fig. 130

The quantities between ( ) are the arc capacities

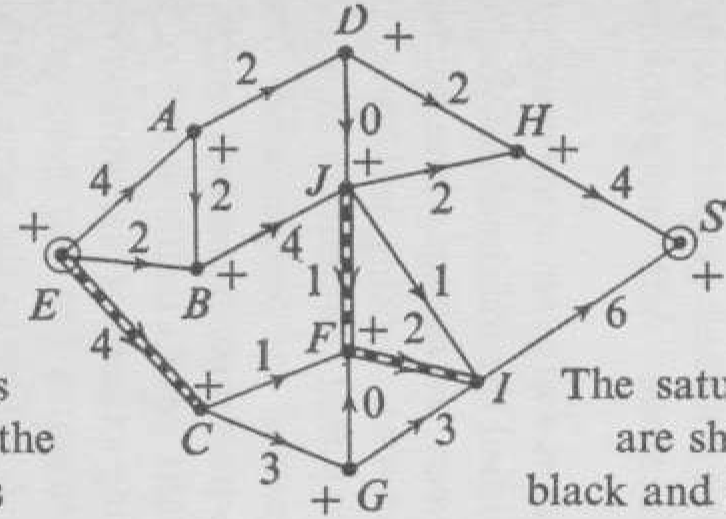


Fig. 131

The saturated arcs are shown as black and white paths

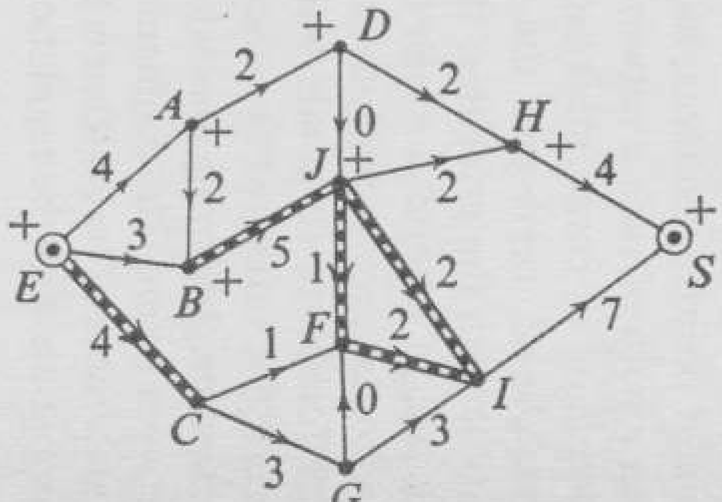


Fig. 132

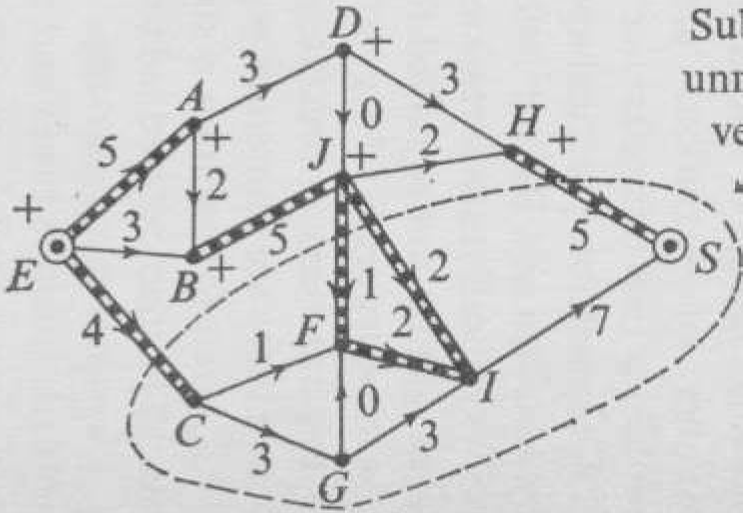


Fig. 133

and those which leave it (there are none in this example) will have a non-zero flow through them.

The example given above relates to an antisymmetric graph, but the Ford-Fulkerson method is useful for all transport networks having or not having antisymmetric graphs. A certain number of variations are concerned with neighbouring problems (see our reference 1).

A remarkable practical application was given some years ago by Matthys and Ricard[2], who studied the maximal delivery of railway goods transportation between two sidings with a double-track section not signalled for two-way working and based on non-priority traffic. It is in this way that the delivery of goods trains has been appreciably improved on certain main lines where there are numerous priority passenger trains and where it is required to maintain a significant goods traffic at the same time. The method used led to solving a transport network where the vertices are *space-time* co-ordinates but the arcs, going towards positive times and corresponding to the same station or the same shed, have a given capacity g_i, whereas the vertical arcs, corresponding to the same instant but going in the direction of positive distances, have a capacity equal to 1. We note that a path of such a network, that is to say an uninterrupted succession of consecutive arcs going from the origin A to the terminus Z, represents the schedule of a goods train compatible with the priorities; the vertical arcs correspond to the passage of a train through the next station or shed, and the horizontal arcs to various stoppings of the train. In fact, the arcs forbidden to goods trains are not only those which can already be saturated by other trains; we take also as saturated certain neighbouring arcs which satisfy the safety requirements between trains. The problem then consists of finding the maximal number of trains (that is to say, of paths) going from A to Z compatible with the regulations imposed. Where the number of

trains and stations can be greater, we often use for such problems a suitably programmed computer.

REFERENCES

1. Ford, L. R., and Fulkerson, D. R., *Flows in Networks*. Princeton University Press, 1962.
2. Matthys, and G. Ricard, M., 'Etude du Débit Maximum entre Deux Tirages'. *Revue Francaise de R.O.*, No. 15, 1960.

9. Important Generalisations of the Concept of 'Graph'

We are concerned here with various concepts whose specifications can vary from one author to another. In fact, none of the concepts considered in what follows deserves the name of *graph* as we defined it at the outset, but, in the theory and applications which we shall consider, the vocabulary is so short of words that we are forced to repeat some words and chance occasional possible misunderstanding. But the reader has been warned!

p-Coloured Graph

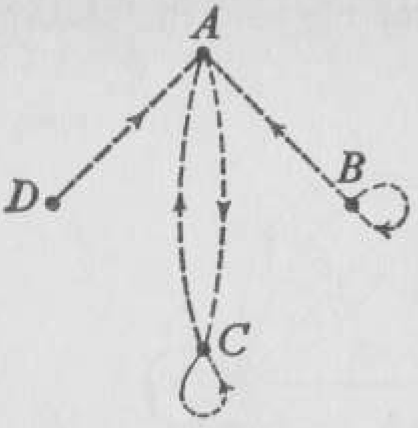
Consider p graphs $G^1 = (\mathbf{E}, \Gamma_1)$, $G_2 = (\mathbf{E}, \Gamma_2)$, $\ldots$, $G_p = (\mathbf{E}, \Gamma_p)$. Let us put together on one diagram, coloured if necessary with different colours for recognition purposes, the arcs of G_1, of G_2, $\ldots$, of G_p. We will have a *p-coloured graph* which we denote by:

$$G = (\mathbf{E}, \Gamma_1, \Gamma_2, \ldots, \Gamma_p).$$

Such a concept arises, for example, in stochastic processes in a countable number of states when the choice of an arc of a graph corresponds to a decision.

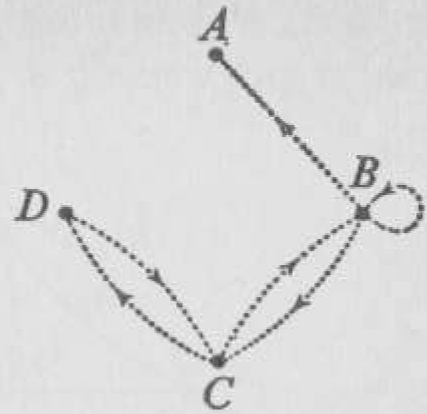
For example, the graphs G_1, G_2, and G_3 of figures 134, 135 and 136 make up the three-coloured graph of figure 137.

The concept of a p-coloured graph has nothing to do, as we explain it, with the concept of a non-oriented r-chromatic graph.



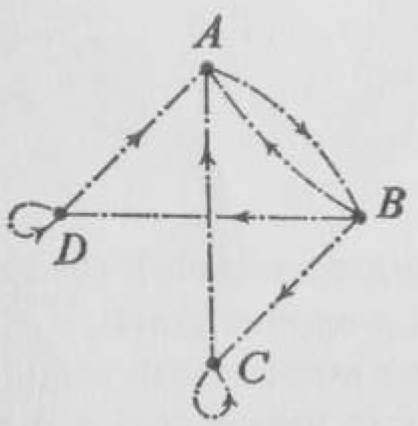
$G_1 = (\mathbf{E}, \Gamma_1)$

Fig. 134



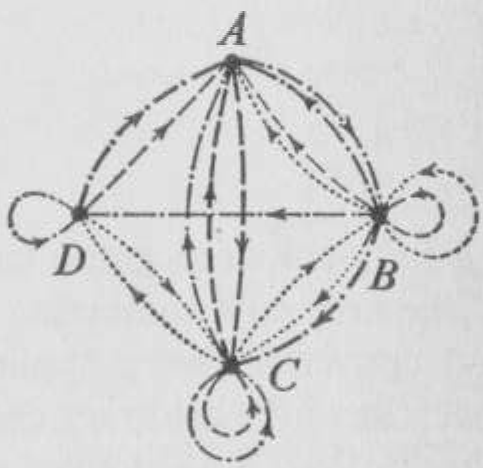
$G_2 = (\mathbf{E}, \Gamma_2)$

Fig. 135



$G_3 = (\mathbf{E}, \Gamma_3)$

Fig. 136



$G = (\mathbf{E}, \Gamma_1, \Gamma_2, \Gamma_3)$

Fig. 137

p-Applied Graph

Let us consider the elements of a finite set forming the vertices of a graph. Let us unite the vertices by arcs in such a way that several arcs having the same sense can connect an ordered pair of vertices X_i and X_j. If the greatest number of arcs in the same ordered pair is p, we have made a *p-applied graph.*

Thus, in the three-coloured graph of figure 137, if we eliminate the differences of colour we will obtain a three-applied graph (figure 138). But it would be easy to find another

example where a q-coloured graph would in this way give a p-coloured graph, where $q \neq p$.

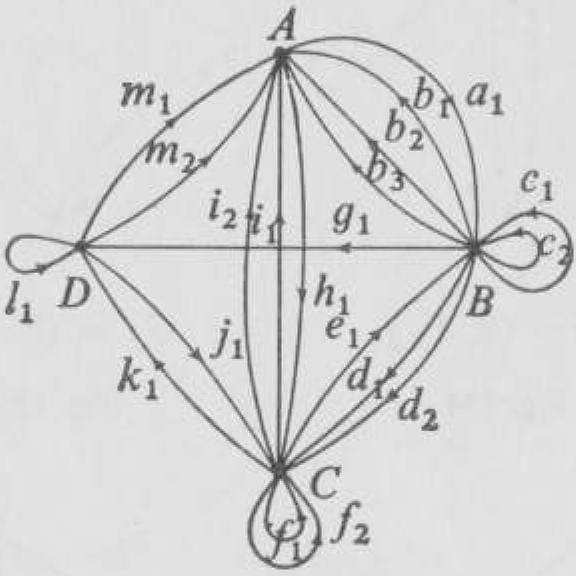


Fig. 138

Any q-coloured graph can non-uniquely establish (because of the arbitrary numbering of arcs) a p-applied graph, $p \leqslant q$, and, conversely, any p-applied graph can non-uniquely establish (because of the arbitrary choice of arcs to be coloured with the same colour) a q-coloured graph, $p \leqslant q$.

The concept of p-applied graph arises in decision problems of a combinatorial nature, in certain communications problems, in the general theory of automata, and so on.

Multigraph or p-Graph

Another generalisation relates to the idea of a non-oriented graph. Given a finite set **E**, each element of which is considered as a vertex, we join by arcs certain vertices X_i to certain others $X_j (X_i \neq X_j)$ in such a way that more than one arc can exist between X_i and X_j. If the greatest number of arcs between a given pair $[X_i, X_j]$ is p, we say that we have a *non-oriented multigraph of order p* or *p-graph*. A p-graph is not a graph in the

sense given at the start of this work, first because it is a non-oriented concept and also because several branches can correspond to the same pair of vertices.

For example, we have shown a four-graph in figure 139.

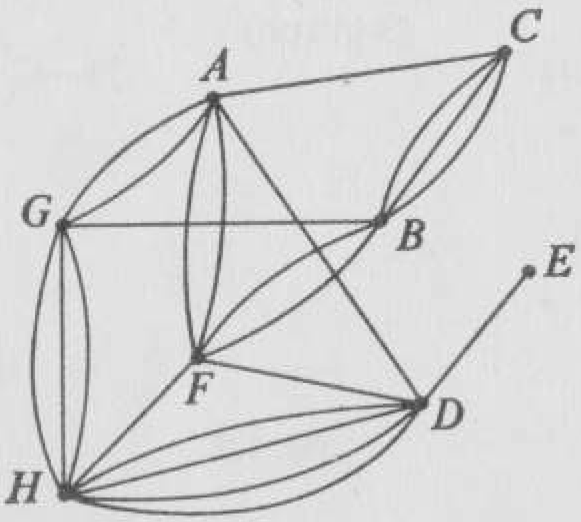


Fig. 139

The concept of p-graph is found in chemistry, sociology, electric networks, and so on.

In figure 140 we have shown an electric network with sources, resistances, inductances and capacitances. The structure of this network (see figure 141) gives a p-graph.

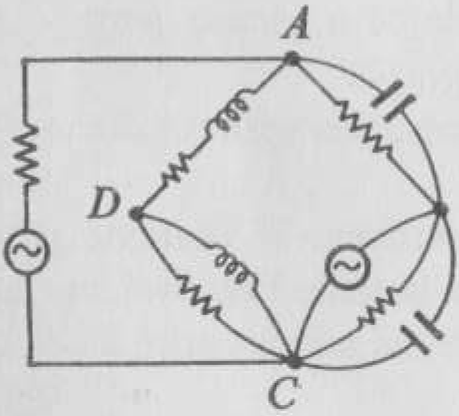


Fig. 140

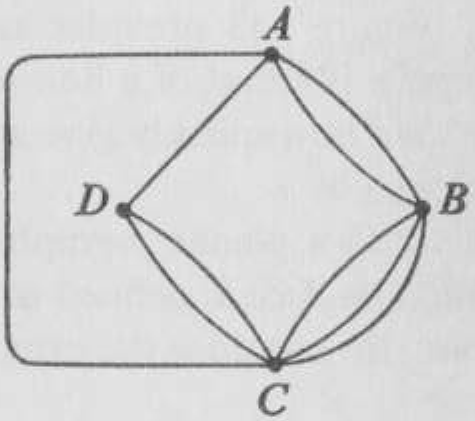
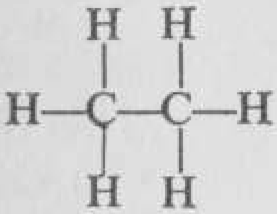


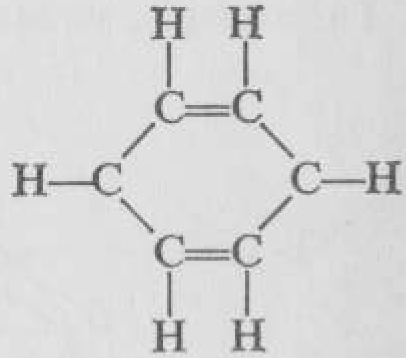
Fig. 141

In figure 142, we have recalled the structure of some organic chemical compounds.



Ethane
(1-graph)

Acetylene
(3-graph)



Benzene
(2-graph)

Fig. 142

Planar p-Graphs

We say that a p-graph is *planar* if it is possible to represent it in a plane in such a way that the vertices are distinct points, the branches are simple curves, and no two branches meet other than at their end-points. We then say that the p-graph is constructable on a plane. This concept has been further extended to constructability on other surfaces (sphere, torus, etc.). We shall consider here only the plane.

Figure 143 provides an example of a planar p-graph and figure 144 that of a non-planar p-graph.

We now quickly give several properties without demonstrations.

1. If a planar p-graph has N vertices, M branches and F faces (a *face* is defined as a region bounded by branches, and we also consider the exterior region as a face), then:

$$N - M + F = 2.$$

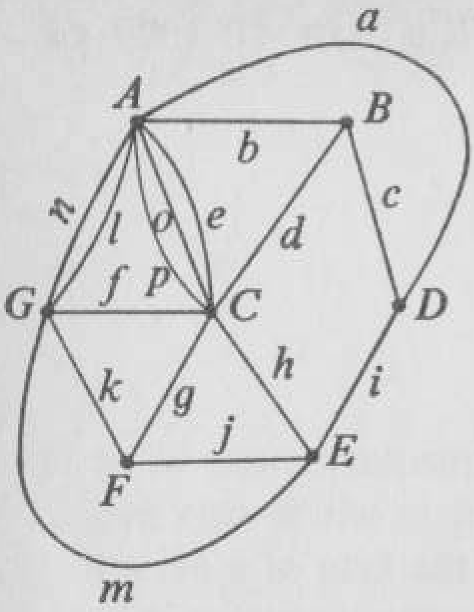


Fig. 143

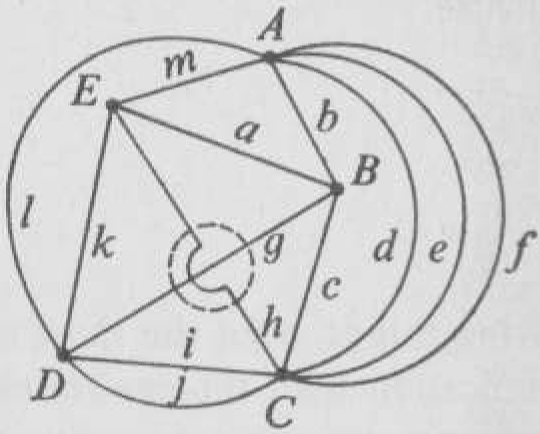


Fig. 144

2. In every planar one-graph, there is a vertex X whose degree is less than or equal to five.

3. Every planar one-graph is five-chromatic. It has never been proved that every planar one-graph is four-chromatic, but we assume this.

The concept of planar graph is of interest in the construction of printed electronic circuits where, as we suspect, the passage of one circuit over another complicates the technology. Such matters are also of interest to motorway constructors, where we want to avoid roads passing over or under each other.

10. From the Concept of 'Graph' to that of 'Image'

Living things, even the most elementary, need to be in communication with the environment in which they evolve. This communication is achieved with the help of a nervous system sometimes reduced to a chemical contact mechanism and, in the case of certain superior species, so complicated that we only know part of its mysteries. The recognition of shapes plays an essential role in organs of sight, for this takes account of the exterior environment of these superior species; we can add to this the recognition of colours and of distances (relief). Certain electronic machines include means of measurement and of inspection which endow them, in some way, with senses adapted to their functions; we have gone further and constructed machines able to recognise sounds and, now, to recognise images or, more precisely, shapes.

The study of the machine recognition of shapes, at the moment still at the outline stage except for the optical scanner, is full of promise and it is interesting to discuss here with the reader the important connections existing between the theory of shape recognition and that of graphs.

If we admit that sight is really a phenomenon of discrete perception (we confirm this by examining the make-up of an eye), each object observed forms a retinal image which is transmitted in a certain code by a cortex. The definition of the image transmitted by the cortex depends for its degree of quality on the device of recognition. Thus, in a 10×10 matrix we can

form a simplified image of a poodle (see figure 145). The black and white image of this poodle is really a graph, but who would recognise this image of a poodle by means of the arrow representation of this graph in figure 146? The same image of the poodle, coded for line-by-line television transmission, appears as shown in figure 147; it is just as difficult to recognise the poodle before the decoding which will reconstitute its image. What is the mechanism which enables instantaneous analysis of the image of figure 145 and the recognition of a poodle there? This mechanism of *global appreciation* is a mechanism of comparison. Images of dogs exist in our memory and our cerebral cortex selects them so as to compare the observed image with the images most resembling it and, finally, by an exchange of information between the cortex and the thalamus the decision to accept or reject the hypothesis is taken. Thus, a greyhound could be accepted or rejected (see figure 148) on comparison with being a poodle; it all depends upon the quality of the definition and on the coincidence instructions pre-existing through experience in the cortico-thalamic organism. Thus, the recognition of shapes is, in some way, an exploration of *visual semantics*. An eye transmits its information simultaneously to the cortex; there is not a single channel as with a television camera which transmits everything 'in Indian file'; there are as many channels as there are sensing points (cones and rod cells, for example).

Meanwhile the 'controllers' or 'super-computers' effect the 'parallel' treatment of information; in general, we do not arrange the actual moment when computers (that is to say electronic calculators) are sequentially dealing with information 'in Indian file'. The way in which the information is coded for transmission also plays an essential role.

If we really wish to detach ourselves from the usual arrow representation of a graph and consider rather its representation

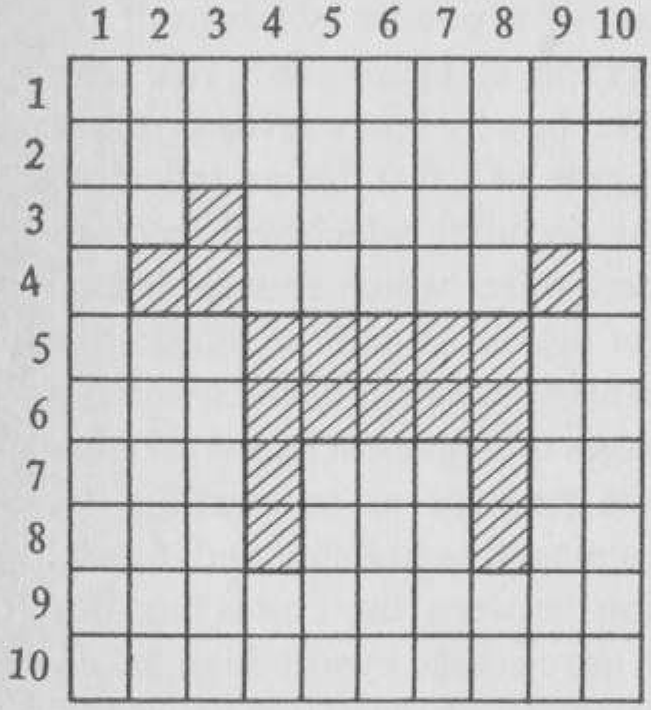


Fig. 145

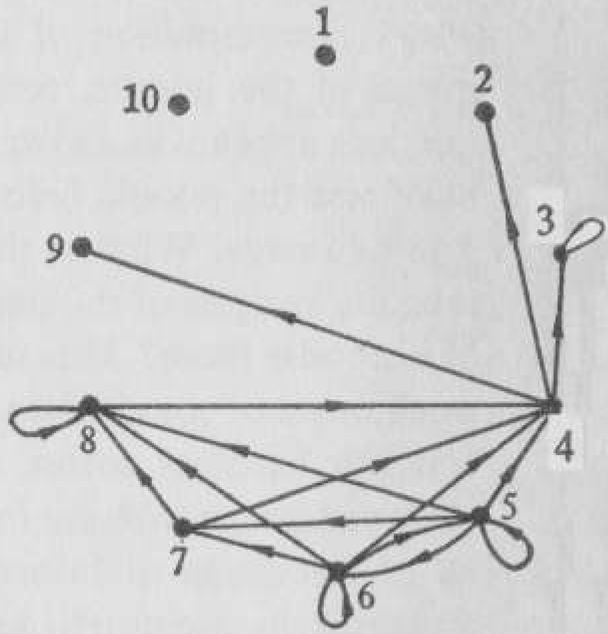


Fig. 146

$$\begin{matrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&1&1&0&0&0&0&0&1&0\\0&0&0&1&1&1&1&1&0&0\\0&0&0&1&1&1&1&1&0&0\\0&0&0&1&0&0&0&1&0&0\\0&0&0&1&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{matrix}$$

Fig. 147

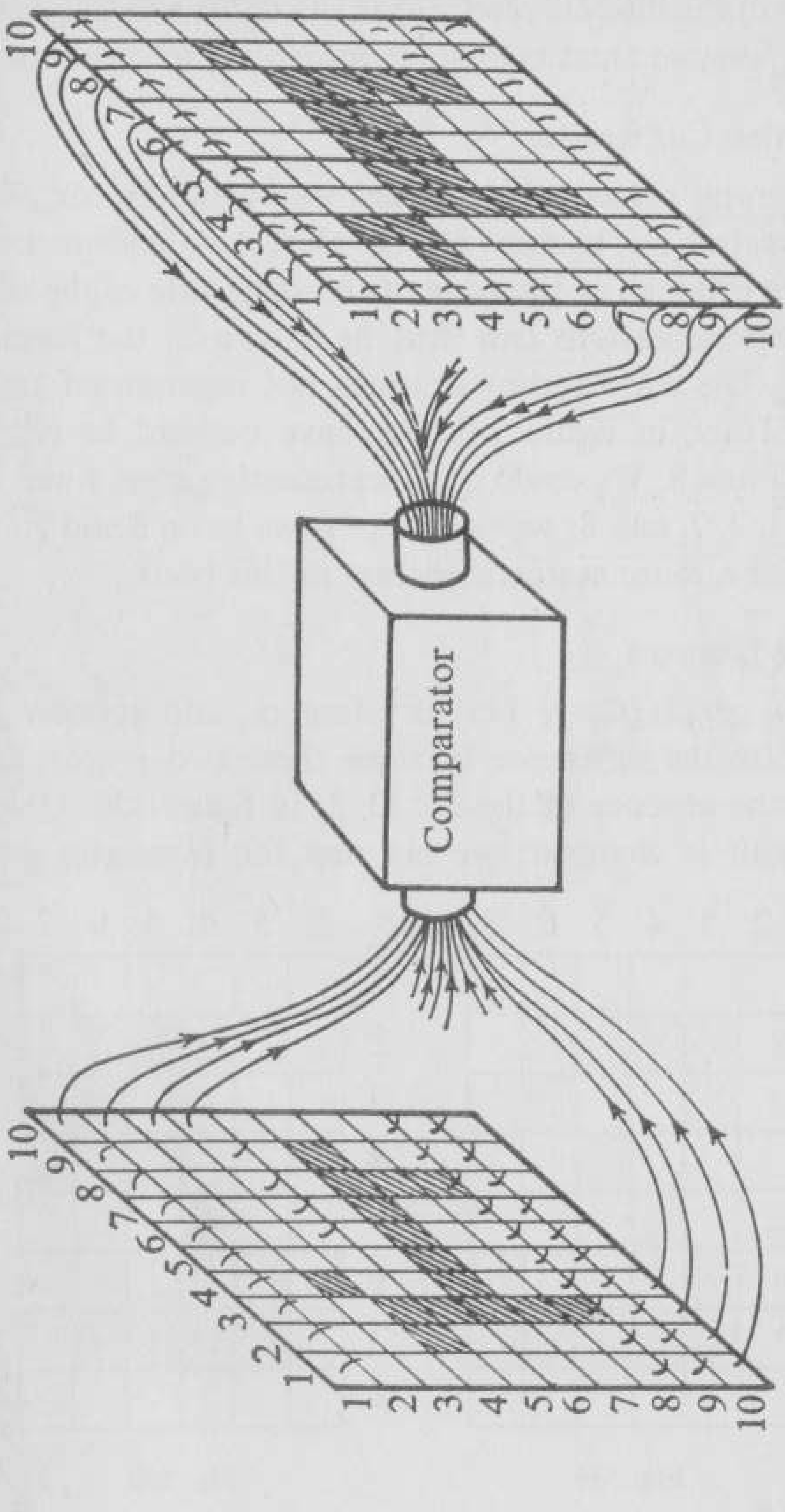


Fig. 148

in a grid or matrix, we see the connection appearing between problems of the discrete recognition of shapes and the theory of graphs, these two areas belonging, moreover, to combinatorics.

Construction Convention

For a graph presented in a black and white matrix, we will allow ourselves not to show all the white rows and/or columns if they are grouped at the edge of the same side of the square. Matrix representations can thus be shown in the form of a rectangle. The rows and/or columns not represented are then implied. Thus, in figure 149, we have omitted to represent columns 1 and 8. We could omit representing rows 1 and 8 and columns 1, 2, 7, and 8; we have kept rows 1 and 8 and columns 2 and 7 for a more aesthetic picture in this book.

Hamming Distance

Given a graph (figure 149) as reference, and another graph (figure 150), the difference between these two graphs resides solely in the absence of the arc (3, 3) in figure 150. Only one ordered pair is changed; we say that the *Hamming distance*

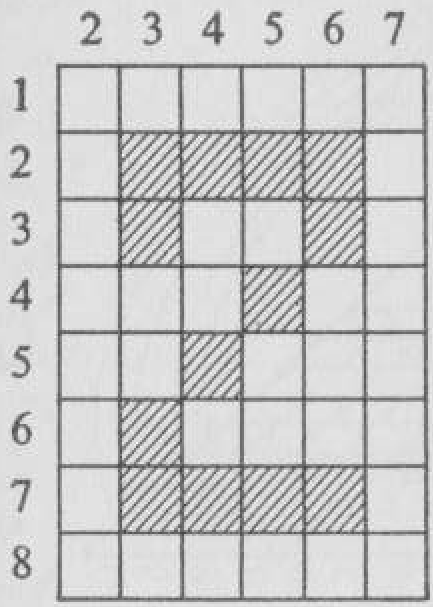


Fig. 149

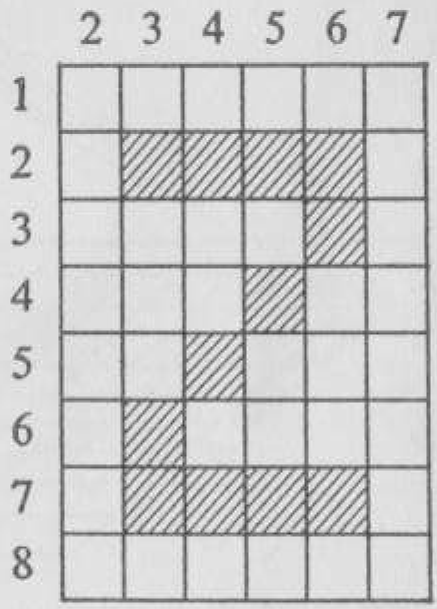


Fig. 150

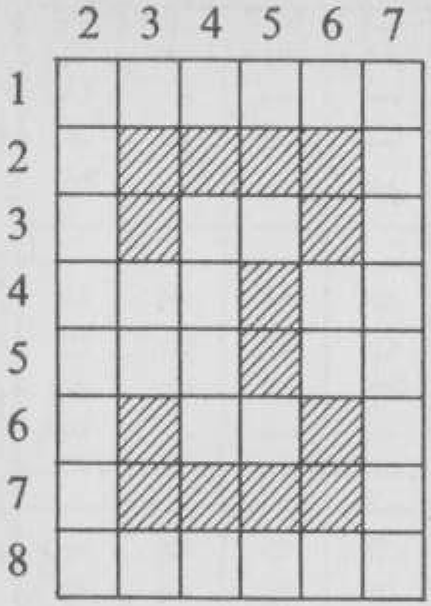


Fig. 151

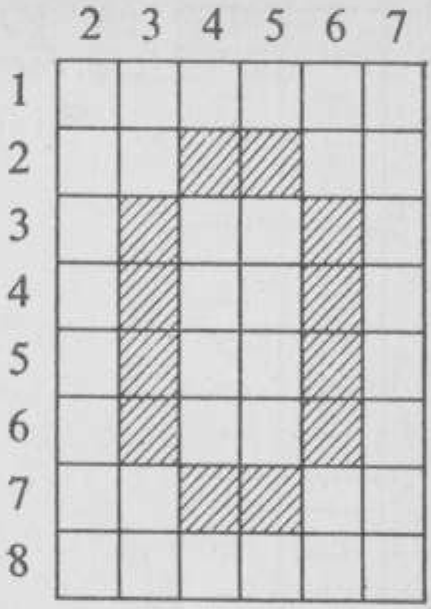


Fig. 152

between the two graphs is 1. Consider now the graphs of figures 149 and 151. To make comparison easier let us write them in the code of figure 147 without taking account of columns 2 and 7 and rows 1 and 8. We see that three ordered pairs have been changed; the Hamming distance between these two figures is 3.

Now, let us represent the respective codes in the four figures 149, 150, 151, 152 and denote them by the figures or letters which we easily read there. Refer to the table of figure 153. Thus, on denoting the Hamming distance between two graphs by $d(G_i, G_j)$, we have:

$$d(2, z) = 1, \qquad d(2, 3) = 3, \qquad d(2, 0) = 11,$$
$$d(z, 3) = 4, \qquad d(z, 0) = 12, \qquad d(3, 0) = 10.$$

Let us recall that a concept of *distance*, such as mathematicians usually consider, has the following properties:

$$d(G_i, G_j) \geqslant 0,$$
$$d(G_i, G_j) = d(G_j, G_i),$$
$$d(G_i, G_j) + d(G_j, G_k) \geqslant d(G_i, G_k);$$

The Hamming distance satisfies these three properties.

Rows	2	3	4	5	6	7
Code for Fig. 149	1 1 1 1	1 0 0 1	0 0 1 0	0 1 0 0	1 0 0 0	1 1 1 1
Code for Fig. 151	1 1 1 1	1 0 0 1	0 0 1 0	0 0 1 0	1 0 0 1	1 1 1 1
				× ×	×	

Rows		2	3	4	5	6	7
Code for Fig. 149	2	1 1 1 1	1 0 0 1	0 0 1 0	0 1 0 0	1 0 0 0	1 1 1 1
— — 150	z	1 1 1 1	0 0 0 1	0 0 1 0	0 1 0 0	1 0 0 0	1 1 1 1
— — 151	3	1 1 1 1	1 0 0 1	0 0 1 0	0 0 1 0	1 0 0 1	1 1 1 1
— — 152	0	0 1 1 0	1 0 0 1	1 0 0 1	1 0 0 1	1 0 0 1	0 1 1 0

Fig. 153

Reception Lattice

In what follows, the word *graph* will be replaced by the word *image*, its being understood that an *image* represented in matrix form with specified quality of definition forms a graph.

Let us consider a matrix of $m \times n$ cells: it is clear that we can define 2^{mn} different images on such a matrix. The set of all possible distinct images constitutes what we call the *reception lattice* of the matrix. Thus the matrix of figure 154 allows

$$2^{5 \cdot 4} = 2^{20} = 1\ 048\ 576$$

definitions.

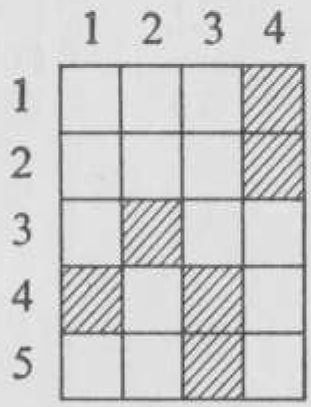


Fig. 154

Below, we give the reception lattices of 1×1, 1×2 (or 2×1), 1×3 (or 3×1), 2×2 (or 1×4 or 4×1) matrices. The figures associated with them are of a particular interest which we are going to describe.

Let us consider, for example, a 2×2 matrix. We say that the 2×2 matrix formed from four black cells is to be the reference set for black cells. This reference includes four elements, A_1 (cell 1, 1), A_2 (cell 1, 2), A_3 (cell 2, 1), and A_4 (cell 2, 2). Let us call the reference $\mathbf{E} = \{A_1, A_2, A_3, A_4\}$. The set of subsets of $\mathbf{E}$ is:

$$\mathscr{P}(\mathbf{E}) = \{\varnothing, \{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_1, A_2\}, \{A_1, A_3\}, \{A_1, A_4\}, \{A_2, A_3\}, \{A_2, A_4\}, \{A_3, A_4\}, \{A_1, A_2, A_3\}, \{A_1, A_2, A_4\}, \{A_1, A_3, A_4\}, \{A_2, A_3, A_4\}, \mathbf{E}\}.$$

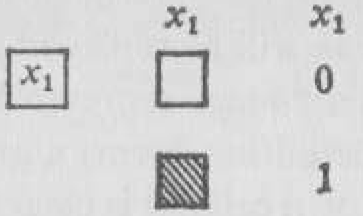
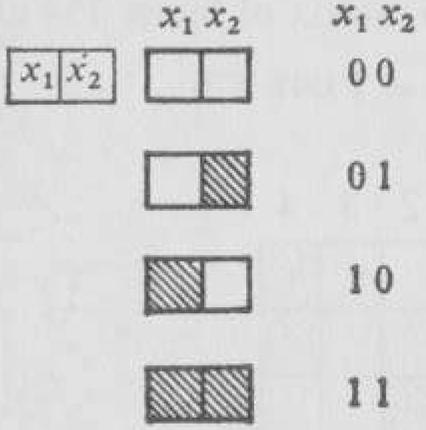


Fig. 155



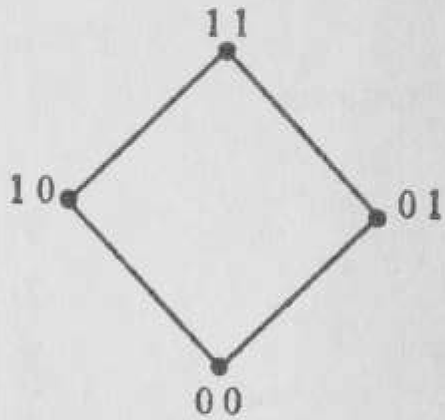
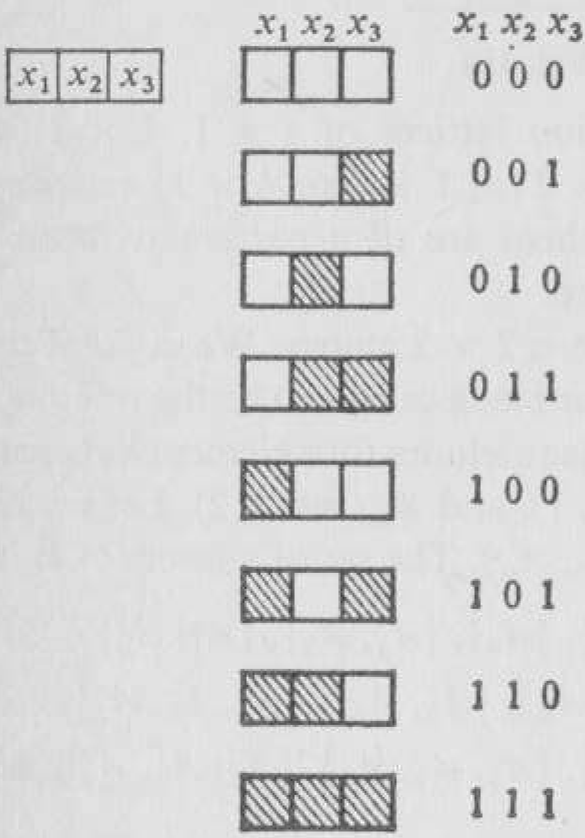


Fig. 156



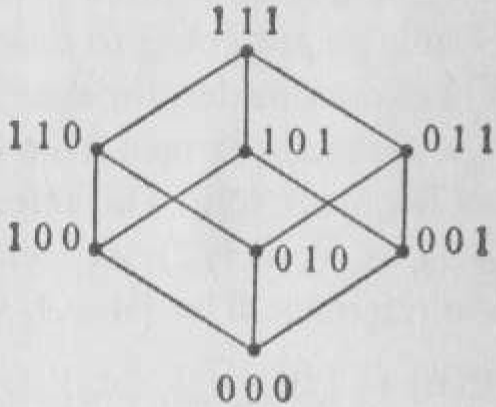
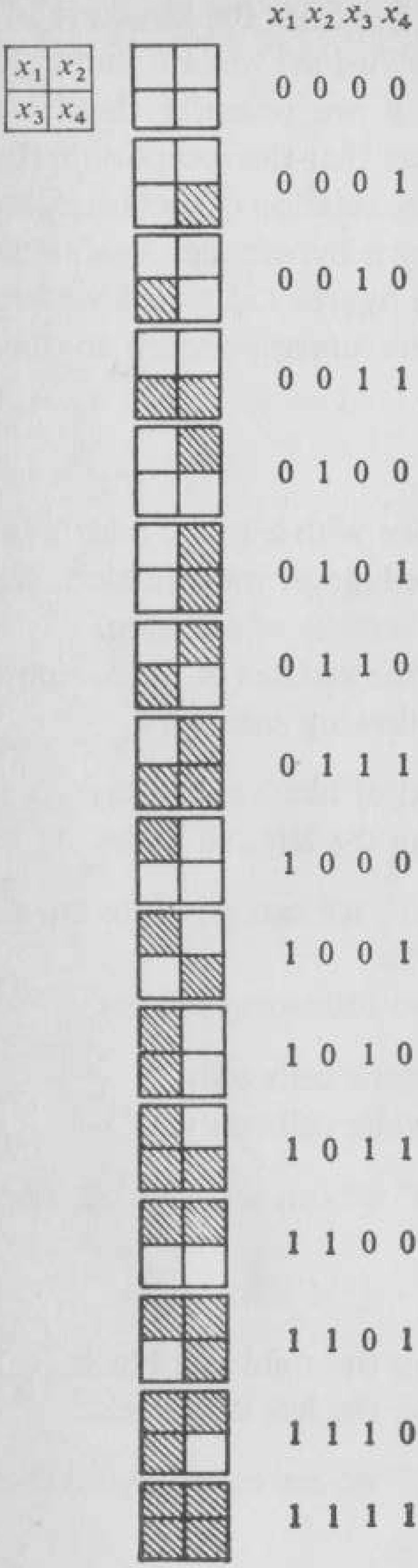


Fig. 157


x_1 x_2 x_3 x_4
x_1 x_2
x_3 x_4
0 0 0 0
0 0 0 1
0 0 1 0
0 0 1 1
0 1 0 0
0 1 0 1
0 1 1 0
0 1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1

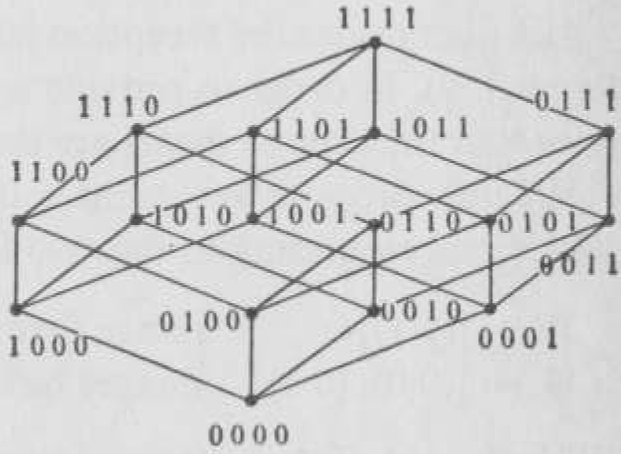

1111
1110
1101
1011
0111
1100
1010
1001
0110
0101
0011
1000
0100
0010
0001
0000

Fig. 158

We know that the set of subsets of a set has the structure of a Boolean lattice for the relation of inclusion; we also know that a Boolean lattice is a hypercube if we consider the Hasse diagram associated with it. It follows that the reception lattice of a matrix is a Boolean lattice for the relation of inclusion, and further that it can be represented by a hypercube. Against the vertices of the hypercubes shown in figures 155 to 158 we have put in binary code the 2^n numbers corresponding to these vertices.

Compactness of an Image

Let us consider the reception lattice with a 1×2 matrix (see figure 159). In order to provide a pedagogic introduction, suppose that the four elements are the vertices of a square.

We now propose to separate out the vertices of the reception lattice corresponding to the two following subsets:

$\mathbf{A} = \{(1\ 1)\}$ image formed of black cells only.
$\mathbf{B} = \{(0\ 0), (0\ 1)\}$ images having the left cell white.

With the aid of a single line Δ or Δ', we can separate out the two subsets **A** and **B** (see figure 159).

In the same way, consider the two following subsets:

$\mathbf{A} = \{(1\ 1)\}$ image formed of black cells only,
$\mathbf{B} = \{(0\ 0)\}$ image formed of white cells only.

With the aid of a single line Δ or Δ'' we can separate out these two subsets (see figure 160).

Again, consider the subsets:

$\mathbf{A} = \{(1\ 1), (0\ 1)\}$ images having the right cell black,
$\mathbf{B} = \{(1\ 1), (1\ 0)\}$ images having the left cell black.

With the aid of a single line Δ' or Δ''' we can separate out these two subsets (see figure 160).

Now, consider the two subsets:

A = {(0 0), (1 1)} images formed by cells of the same kind,
B = {(0 1), (1 0)} images formed by cells of different kinds.

This time, we find that we cannot effect the separation with a single line; we need two lines, for example Δ_1 and Δ_2 (see figure 161).

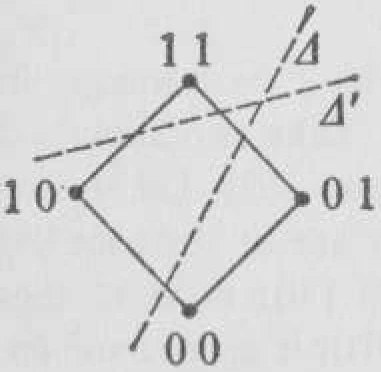


Fig. 159

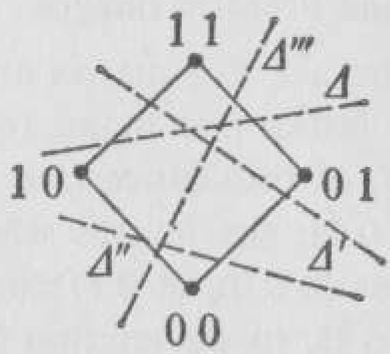


Fig. 160

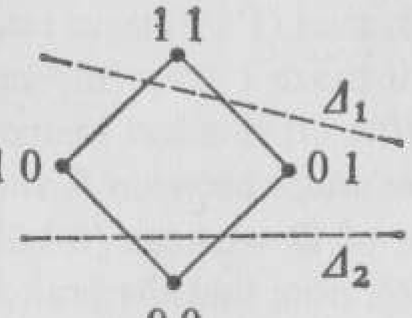


Fig. 161

We can put the same problem when considering this time the case of a reception lattice for more complicated matrices. For example, for 1×3 matrices it is necsary to use separating, planes, for 2×2 matrices separating hypersurfaces. But, in returning to the case of the 1×2 matrix, we see that the separation of certain subsets requires one line only, although the separation of certain others requires two. We say that the

subsets which are the most easily separable (one line only) are more 'compact' than those which are less easily separable (two lines).

Thus, the two subsets {(1 1)} and ({0 0)} are more compact one in comparison with the other than the two subsets {(0 0), (1 1)} and {(0 1), (1 0)}.

Interior and Frontier Images

Consider a 1×3 matrix and its $2^3 = 8$ images forming its reception lattice (see figure 162). Take two subsets of images, **A** and **B**, of this lattice (see figure 162). Let us consider the image (1 0 0); the images which are at distance 1 from this image are: (0 0 0), (1 0 1) and (1 1 0); none of these images belongs to **B**, so we say that (1 0 0) is an *interior image* of **A**. In the same way, we find at a distance 1 from another point (1 1 0) of **A** the points (1 0 0), (0 1 0), (1 1 1) and none of these images belongs to **B**, then (1 1 0) is an interior image of **A**. On the other hand, at distance 1 from the image (1 1 1) we have (1 1 0), (1 0 1) and (0 1 1); this last image belongs to **B**, so we say that it is a *frontier image* between **A** and **B**. Similarly, (0 0 1) is an interior image of **B** whereas (0 1 1) is a frontier image between **A** and **B**. We note that the branches of the hypercube give us images with distance 1 between them.

Let us now give the general definitions of these concepts. Consider two disjoint subsets, **A** and **B**, of images in the same reception lattice. An image element of **A** is *interior* to **A** if it is not at distance 1 from any image element of **B**. If it is at distance 1 from an image element of X, it is a *frontier image* of **A** and **B**.

If we want to separate out the subsets **A** and **B** of figure 162, the one in comparison with the other, this will be easy, There is only one frontier point for one subset with respect to the other

(a plane is sufficient), so we say that the subset **A** of images is 'compact' with respect to **B** (or vice versa).

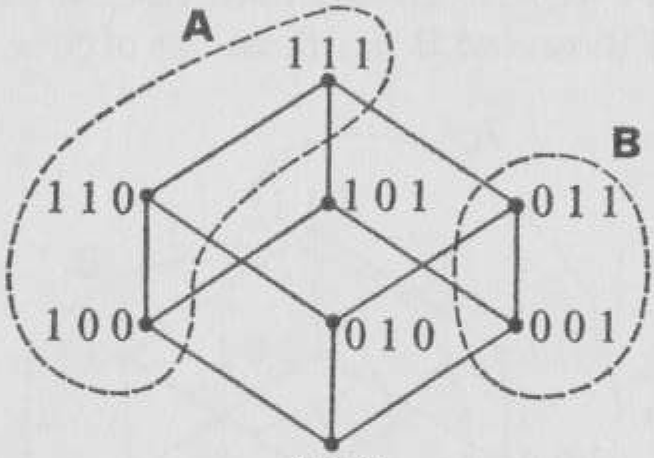


Fig. 162

Take again the example of the 1×3 matrix and its 2^3 images. Choose different subsets **A** and **B** (see figure 163). All the images of **A** are frontier images (the same for all the images of **B**). We say that **A** is not compact with respect to **B** or, in this case also, that **B** is not compact with respect to **A**. For the moment, we reserve a qualitative aspect in the notion of com-

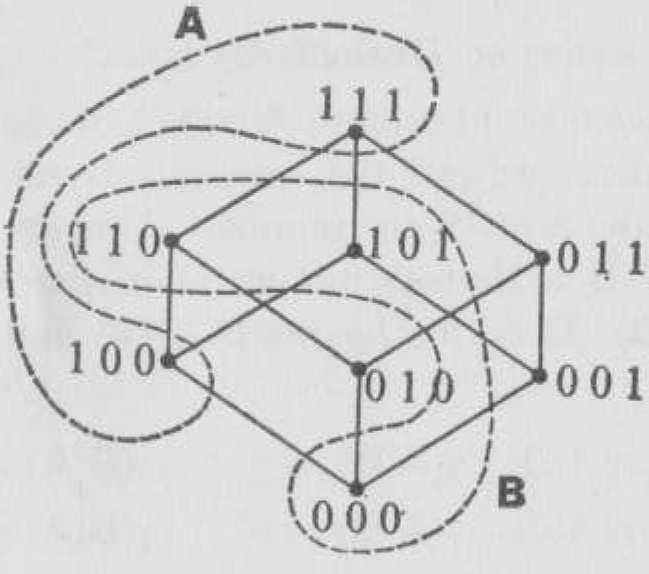


Fig. 163

pactness; we shall later see how to introduce a quantitative aspect.

In figure 164 we have given a third case. **A** has two frontier images out of three and **B** has three out of four.

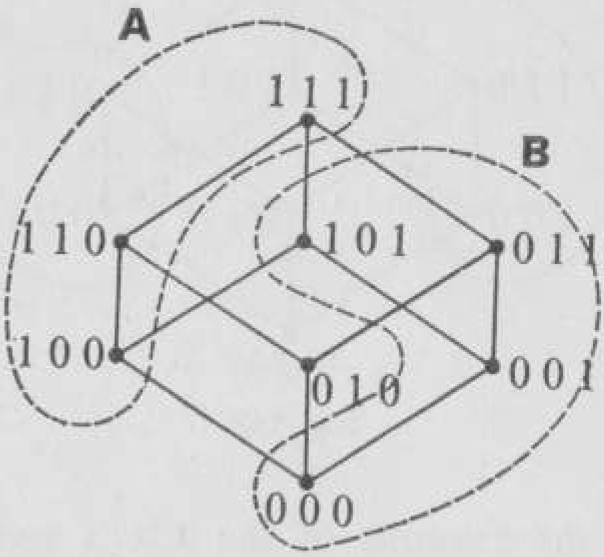


Fig. 164

But we can consider compactness as being qualified by the number of interior images of the subset being considered. We can say that a subset is all the more compact if it has a considerable proportion of interior images.

Compactness Rating or 'Dissimilarity Index'

Consider two disjoint subsets, **A** and **B**, of images of one and the same matrix; we call the *compactness rating* of **A** with respect to **B** the ratio of the number of images of **A**, interior with respect to **B**, to the total number of images of **A**. We denote this by $\gamma(\mathbf{A}|\mathbf{B})$. Thus, for figures 162, 163 and 164, we have respectively:

Figure 162: $\gamma(\mathbf{A}|\mathbf{B}) = \frac{2}{3}$, $\gamma(\mathbf{B}|\mathbf{A}) = \frac{1}{2}$,
Figure 163: $\gamma(\mathbf{A}|\mathbf{B}) = 0$, $\gamma(\mathbf{B}|\mathbf{A}) = 0$,
Figure 164: $\gamma(\mathbf{A}|\mathbf{B}) = \frac{1}{3}$, $\gamma(\mathbf{B}|\mathbf{A}) = \frac{1}{4}$.

We can state: a subset of images is more 'separable' than another when its compactness rating is nearer to 1. We say that a subset of images is *completely compact* with respect to another if its compactness rating is equal to 1. We say that it is *non-separable* if its rating is equal to 0.

Two-colour Images with Gradation

Let us now consider a matrix in which the images can be regarded as being made up of white cells, black cells, and cells having gradations towards black between black and white. Suppose that this gradation is limited to a single intermediate stage, for example a particular grey. In the coding we make a 0 correspond to a white cell, $\frac{1}{2}$ to a grey cell, and 1 to a black cell. Thus, in the code figure 165 gives:

$$1\,\tfrac{1}{2}\,\tfrac{1}{2}\,0\,0\,1\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\,1, \quad 0\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,0\,0\,1,$$
$$0\,\tfrac{1}{2}\,1\,0\,0\,0\,\tfrac{1}{2}\,1\,0\,1, \quad 0\,\tfrac{1}{2}\,1\,1\,1\,0\,0\,1\,0\,1, \ldots.$$

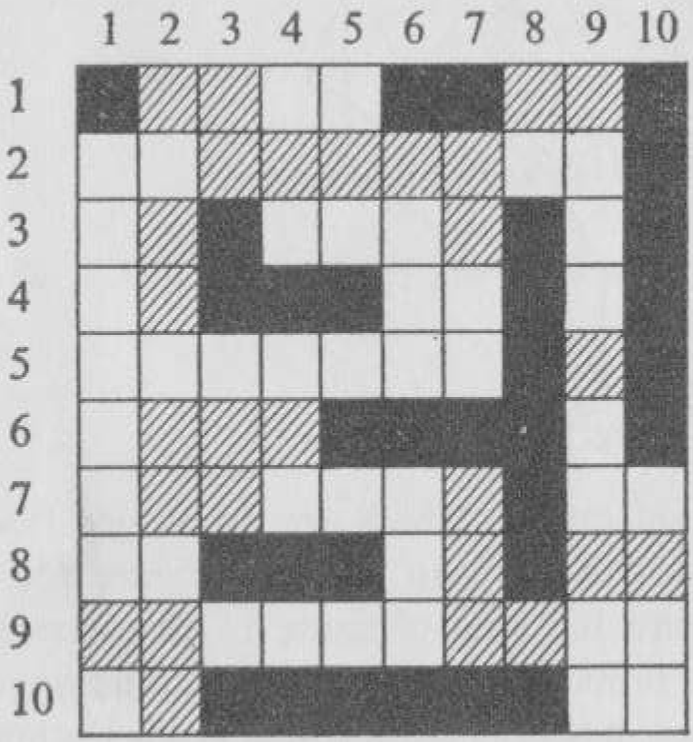


Fig. 165

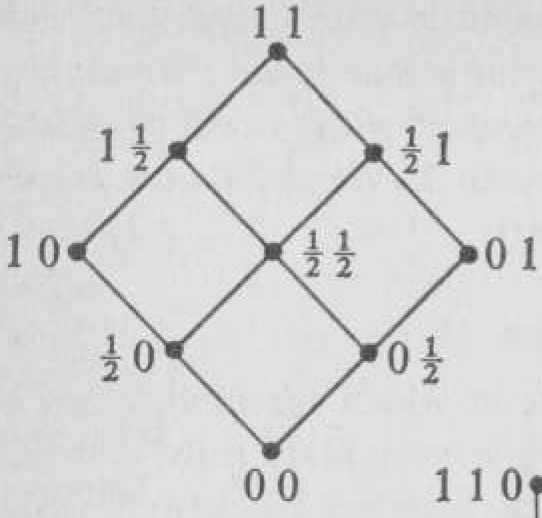


Fig. 166

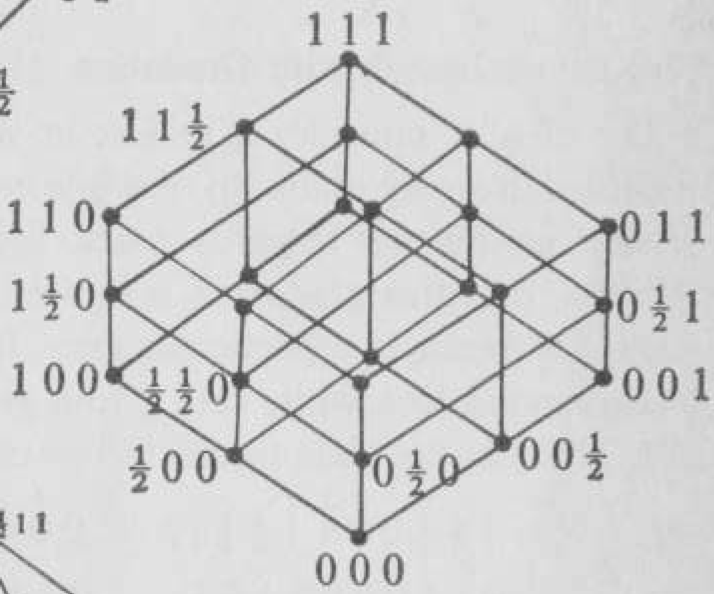


Fig. 167

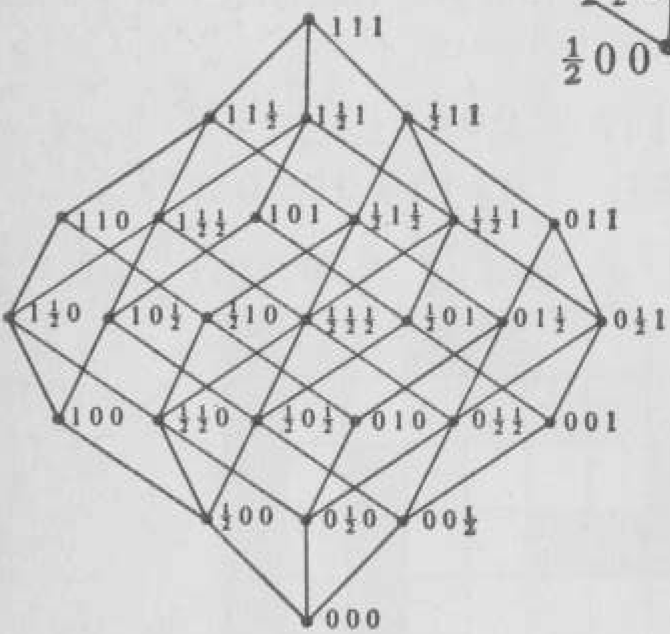


Fig. 168

We then get lattices which are no longer Boolean lattices (hypercubes). The Boolean lattice of figure 156 becomes the lattice of figure 166, that of figure 157 becomes that of figure 167 which is presented more conveniently in figure 168.

In these new lattices we can still define a Hamming distance. Thus:

$$d(0\,\tfrac{1}{2}\,\tfrac{1}{2},\,1\,1\,\tfrac{1}{2}) = 1\tfrac{1}{2}, \qquad d(0\,1\,0,\,1\,1\,1) = 2.$$

In the lattices as represented by their Hasse diagrams, the branches represent Hamming distances equal to $\frac{1}{2}$.

We can generalise this gradation from white to black further by introducing a coefficient $\alpha (0 \leqslant \alpha \leqslant 1)$ representing a continuous gradation from white 0 to black 1. The lattice so obtained is then presented in hypercube form but all the intermediate values of the components $(\alpha_n \alpha_{n-1} \ldots \alpha_1 \alpha_0)$ $0 \leqslant \alpha_i \leqslant 1$ can be taken. If we consider a square (see figure 169) or a cube (see figure 170), the interior and the sides of the square, the interior and the surface of the cube, correspond then to images.

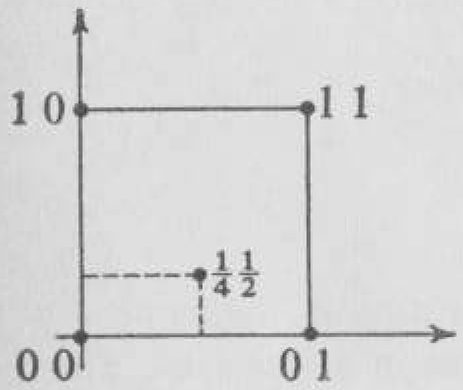


Fig. 169

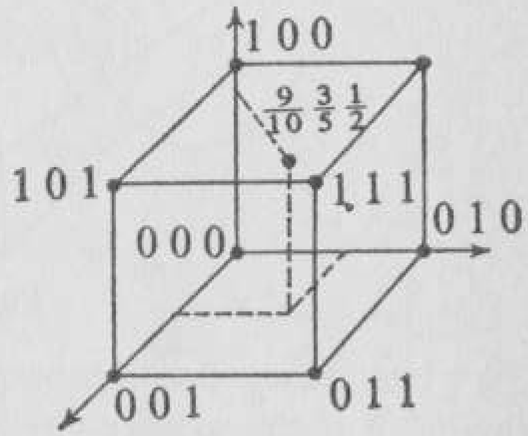


Fig. 170

The Hamming distance between two images

$$(\alpha_n \alpha_{n-1} \ldots \alpha_1 \alpha_0) \quad \text{and} \quad (\alpha'_n \alpha'_{n-1} \ldots \alpha'_1 \alpha_0)$$

will then be expressed by:

$$d(\alpha_n \alpha_{n-1} \ldots \alpha_1 \alpha_0, \quad \alpha'_n \alpha'_{n-1} \ldots \alpha'_1 \alpha_0) = |\alpha_n - \alpha'_n| + |\alpha_{n-1} - \alpha'_{n-1}| + \ldots + |\alpha_1 - \alpha'_1| + |\alpha_0 - \alpha'_0|.$$

This generalised Hamming distance can take all the values

$$0 \leqslant d \leqslant n + 1$$

For example:

$$d(\tfrac{1}{2}\ \tfrac{5}{7}\ 0\ \tfrac{3}{8},\ 1\ 0\ \tfrac{2}{5}\ \tfrac{1}{2}) = \left|\tfrac{1}{2} - 1\right| + \left|\tfrac{5}{7} - 0\right| + \left|0 - \tfrac{2}{5}\right| + \left|\tfrac{3}{8} - \tfrac{1}{2}\right| = \tfrac{1}{2} + \tfrac{5}{7} + \tfrac{2}{5} + \tfrac{1}{8} = 1\tfrac{207}{280}.$$

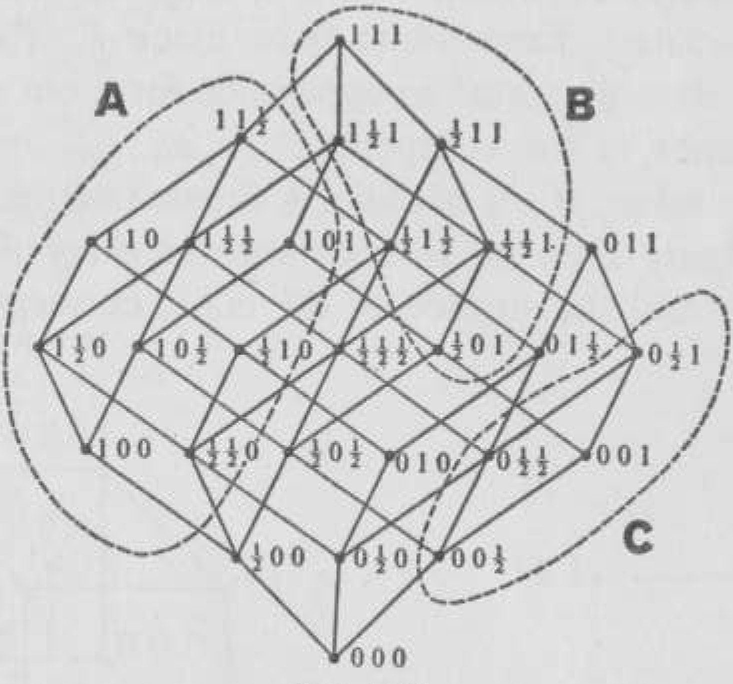


Fig. 171

We can define the compactness rating in the same way as previously if the α_i take their values in a finite set. Thus, in looking at figure 171:

$$\gamma(\mathbf{A}|\overline{\mathbf{A}}) = \tfrac{2}{9},\qquad \gamma(\mathbf{A}|\mathbf{B}) = \tfrac{5}{9},\qquad \gamma(\mathbf{A}|\mathbf{C}) = 1,$$

$$\gamma(\mathbf{B}|\overline{\mathbf{B}}) = 0,\qquad \gamma(\mathbf{B}|\mathbf{A}) = \tfrac{2}{6} = \tfrac{1}{3},\qquad \gamma(\mathbf{B}|\mathbf{C}) = \tfrac{4}{6} = \tfrac{2}{3},$$

$$\gamma(\mathbf{C}|\overline{\mathbf{C}}) = 0,\qquad \gamma(\mathbf{C}|\mathbf{A}) = 1,\qquad \gamma(\mathbf{C}|\mathbf{B}) = \tfrac{2}{4} = \tfrac{1}{2}.$$

The dissimilarity existing between **A** and **C** is visible to the eye (see figure 172).

Practical Uses of Such Concepts

There is enough material now to write several works on this subject (see the reference 1 at the end of this chapter)—we have simply wished to whet the appetite of our reader. In fact, very active research is being conducted on these questions but these,

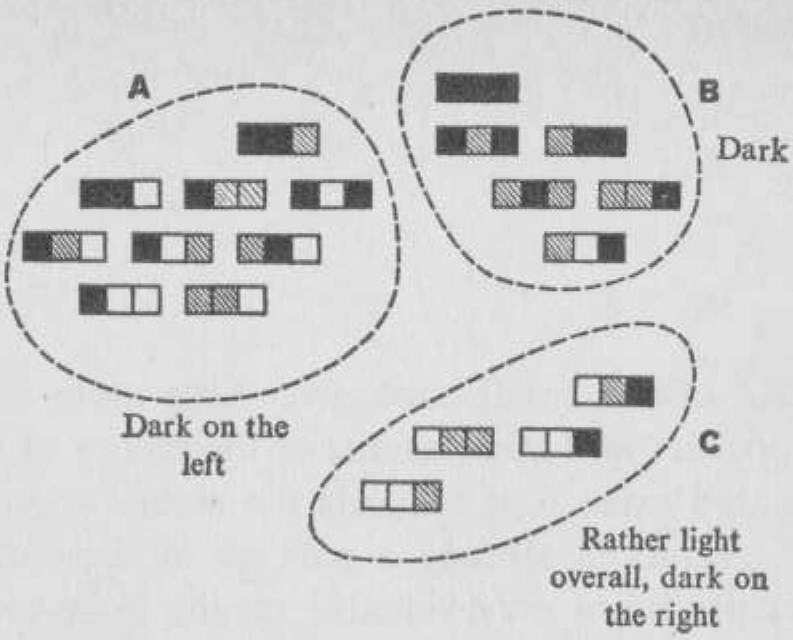


Fig. 172

all too often, remain secret; indeed, the problems which relate to pattern recognition are of great interest to the military for their defence systems. Computers are very slow and inefficient machines when it comes to highly complex images where resources more like the brain than servo-control mechanisms are needed.

New mathematics is in the process of being born and of developing: n-ary logic ($n > 2$), non-associative algebras, Ω-algebras, similarity lattices, and so on. It is still difficult for us to give references for these concepts, the papers are rarely published but this omission will be quickly filled for we conceive that certain highly combinatorial problems cannot remain long without solution if they sufficiently concern responsible politicians, administrators or militarists.

These questions supply excellent thesis subjects for young scientists in every country.

REFERENCES

1. Arkadev, A. G., and Braverman, E. M., *Teaching Computers to Recognize Patterns*. Academic Press, New York, 1967.

Conclusion

The content of this small book gives little more than an idea of the properties and applications of the theory of graphs and its anticipated extensions towards the *theory of patterns*. The theory of graphs is already a part of modern mathematics which has produced very valuable results in several fields of pure and applied science and also in the arts: economics, military strategy, industrial organisation, communications networks, crystallography, organic chemistry, sociology, psychology, aesthetics, languages, programming, textual criticism, musical composition, algorithmic painting, analysis of psycho-sociological situations, and so on.

The peculiar domain of the theory of graphs overlaps others such as Boolean algebra, lattice theory, linear programming, number theory, combinatorics, electric network theory, flow graphs, etc.; the list is not finished.

No engineer, or physicist, or chemist can ignore this theory; otherwise, complicated structures will become for them labyrinths which lie outside scientific method. Economists and strategists recognise better the nature of the phenomena in which they are interested if the structures are presented from the start as graphs. Ultimately, it is the set of combinatorial phenomena which is the concern of the theory of graphs.

The engineer and the artist will be brought close together by the theory of graphs and its extensions; will it not be a good thing for them to join together in logical and global understanding and thus be more completely human?

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(Note: Where a word is defined and then subsequently used frequently in the book, only the reference page for its definition is given.)